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Lie pseudogroups and differential sequences

New perspectives in two-dimensional conformal geometry *

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By introducing the general notion of Lie pseudogroups and related geometrical concepts, it is shown that all affine and projective structures on a Riemann surface emerge from the computation of certain differential invariants for a suitable differential sequence, namely the non-linear Janet sequence for the 2-d conformal pseudogroup.

Similarly the corresponding gauge theory (for the same 2-d conformal pseudogroup) is related to the first non-linear Spencer sequence. It is proved that the latter incorporates exactly the vanishing curvature conditions obtained in a BRS differential algebraic framework by L. Baulieu, M. Bellon and R. Grimm (Phys. Lett. B 260 (1991) 63) for the "gauging of the Virasoro algebra". A proper geometrical meaning of this reference is thus provided according to our jet theoretical formulation.

At the same time the cohomological link between the non-linear Janet sequence and the first non-linear Spencer sequence is explained.

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1. Introduction

For a few years now, an important part of the activity in field theory has been devoted to the study of the quantization of low dimensional fields subject to a gauge symmetry. It is fair to say that most quantization schemes that are used are

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of a geometric type, namely quantization over moduli spaces. Besides the fact that these quantization procedures have been commonly preferred, a *Lagrangian formulation* of bidimensional conformal field theories seems to emerge from only *two different geometric* approaches both related to a differential algebra implemented through the BRS operation [1,2].

Let us briefly say that the first one, developed in refs. [3–6], using *concepts of pure differential geometry* (atlas, manifolds, structures, bundles, etc.) provides a satisfactory well-definedness on a Riemann surface, while the second, elaborated in refs. [7–9], appeals to the useful *moving frame* technique with covariance insured through the use of differential forms.

Moreover, on the one hand there is a tentative construction [10] using differential geometry for treating the so-called W-algebras by seeking after purely classical (differential geometric) objects. On the other hand, by a pure analogy with the BRS differential algebraic construction for Yang–Mills theories, some algebraic equations generating both the maximal Lie subalgebra (denoted w_2) of the Virasoro algebra and the so called $w_{1+\infty}$ algebras are exhibited in ref. [11]. About the gauging of the Virasoro algebra see also ref. [12].

In the light of the introduction of ref. [10], the notion of jet bundles and flag configurations seems to occur at some point in the formalism and it would be nice to be able to go forward along this line. Moreover as stated in the introduction of ref. [11], the *idea* of starting first with a *given symmetry group* (or a given Lie algebra) rather than looking for all possible symmetries of an *a priori* Lagrangian and then gauging its Lie algebra (or the given Lie algebra) thanks to the BRS technique might be *pertinent* at the conceptual level.

Having this in mind, we shall introduce in the present paper a classical tool, which will be the formal theory of PDEs developed by Spencer and collaborators during the period 1963–1975 [13–15] for the study of finite length differential sequences that can be constructed for *Lie pseudogroups.* (A Lie pseudogroup is a group of transformations which are solutions of a system of PDEs.)

The application of this formal theory will be restricted to the case of 2-d conformal field theories. The purpose of this work is not to present the complete and rigorous theory but to show that *there exists a general framework* within which a variety of results concerning 2-d conformal field theories may be placed. In dealing with it, we shall observe that, instead of being directly confronted with an *infinite dimensional* group of transformations, it is possible to recover a *finite* dimensional situation thanks to such concepts as Lie groupoids and Lie algebroids, which will generalize those of (finite) Lie groups and (finite) Lie algebras.

The paper is organized as follows. In section 2 we shall summarize the necessary ingredients (jet theory, groupoid theory, differential invariants) for the construction of the non-linear Janet sequence [16] and the first non-linear Spencer sequence [13] for the case of Lie pseudogroups, together with the intricate link existing between them. Due to the lack of space and the technical nature of many arguments the reader will be supposed to be familiar with a few basic definitions that can be found in textbooks, e.g., refs. [17,18]; for a concise review see, e.g., ref. [19]. However, this will fix the notation and we shall insist on the more specific concepts and methods according to the presentation adopted by the second named author in dealing with applications to mathematical physics [20–22]; for a short review of ref. [22] see ref. [23].

Section 3 is devoted to the construction of the non-linear Janet sequence for the 2-d complex analytic (conformal for short) pseudogroup. In this framework differential invariants and their related geometric objects for the complex affine and projective structures are computed. In doing so with respect to our formulation, we recover among other things the proper geometrical context which served as a starting point for a recent work "On the origin of W-algebras" [10].

Section 4 deals with the construction of the first non-linear Spencer sequence and its projective limits for the conformal pseudogroup. The analysis of gauge variations is provided. Thanks to the jet formulation thereby introduced the first part of ref. [11] is reinvestigated in this "non-standard" approach.

Three appendices are added where some computational details are gathered.

2. Lie pseudogroups and differential sequences

Throughout this work, manifolds and maps will be supposed to be smooth. We shall use the *same* notation for bundles and sheaves of germs of sections whenever the context is clear: an operator will be an arrow between two sheaves of germs of sections.

2.1. JANET SEQUENCES

Let X be a real (oriented) manifold of dimension n with local coordinates $x = (x^i)$, $i = 1,\dots,n$. We shall denote by $T = T(X)$ and $T^* = T^*(X)$ the tangent and cotangent bundles of X, respectively, and we shall use the symbols $\otimes$, S, $\wedge$, for tensor, symmetric and exterior products. Let $J_q(X,X)$ be the bundle of q-jets of maps from X to X, $f : X \to X$, *identified with their graph* $\tilde{f} : X \to X \times X$, $x \mapsto (x, f(x))$ as sections of the trivial bundle $X \times X$ over X equipped with the *source projection* α onto the first factor and the *target projection* β onto the second factor. We have $\alpha \circ \tilde{f} = \mathrm{id}_X$. From now on, we shall write $\tilde{f} \equiv f$.

The local coordinates on $J_q(X,X)$ will be denoted by (x^i, y^k_μ), $i,k = 1,\dots,n$, or simply (x, y_q), with $y^k_0 = y^k$ and where the multi-index $\mu = (\mu_1,\dots,\mu_n)$ has length

$$0 \le |\mu| = \mu_1 + \cdots + \mu_n \le q \,. \tag{2.1}$$

We also set

$$\mu + 1_i = (\mu_1, \ldots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \ldots, \mu_n)\,. \tag{2.2}$$

The transition functions of the jet bundle are provided by those of derivatives

$$\partial_\mu f^k(x) \equiv \partial^{|\mu|} f^k(x)/(\partial x)^\mu$$

by replacing derivatives by jet coordinates y^k_μ. The fiber dimension of $J_q(X,X)$ is $n\binom{n+q}{q}$. There are canonical projections

$$\pi^{q+r}_q : J_{q+r}(X,X) \to J_q(X,X),\ (x, y_{q+r}) \mapsto (x, y_q), \quad \forall q, r \geq 0, \tag{2.3}$$

which correspond to truncations at order q of the Taylor expansion up to order $q + r$.

We shall denote by $\Pi_q(X,X)$ or simply Π_q the open subbundle of $J_q(X,X)$ defined by the local condition $\det(y^k_i) \neq 0$. So Π_q is the bundle of q-jets of invertible maps from X to X. The corresponding source and target projections of Π_q on X are denoted by $\alpha_q = \alpha \circ \pi^q_0$ and $\beta_q = \beta \circ \pi^q_0$, respectively.

We shall usually denote a *section* of Π_q over f by $f_q : X \to \Pi_q,\ x \mapsto (x, f^k_\mu(x))$ such that $\alpha_q \circ f_q = \mathrm{id}_X$ and $\pi^q_0 \circ f_q = f$. To any section f of $X \times X$ there corresponds the section $j_q(f) : X \to \Pi_q,\ x \mapsto (x, \partial_\mu f^k(x))$ of Π_q (the q-jet of f). We can summarize the situation in the following diagram:

$$\begin{tikzcd} \Pi_q \arrow[r, "\pi^q_0"] \arrow[d, "\alpha_q"] & X \times X \arrow[d, "\beta"] \\ X \arrow[u, shift left=3, "{f_q,\ j_q(f)}"] \arrow[ur, "\tilde{f} \equiv f"'] \arrow[r, "f"] & X \end{tikzcd}$$

Remark 2.1. In general $f^k_\mu \neq \partial_\mu f^k$. This distinction is crucial as will be seen later on.

We recall that a *Lie groupoid* of order q on X is a fibered submanifold $\mathcal{R}_q \subset \Pi_q$ with composition law $\mathcal{R}_q \times_X \mathcal{R}_q \to \mathcal{R}_q$ and inverse $\mathcal{R}_q \to \mathcal{R}_q$ induced by those for Π_q that can be defined through the well known chain rule for derivatives with jets instead of derivatives. Here the fibered product must be taken with respect to the *target* projection β_q on the left and to the *source* projection α_q on the right, and the inverse by reversing source and target. The *identity* element is given by $\mathrm{id}_q = j_q(\mathrm{id}_X)$, the q-jet of the identity map $\mathrm{id}_X : X \to X \times X,\ x \mapsto (x,x)$. Let us introduce the *Lie algebroid*

$$\mathrm{R}_q = \mathrm{id}_q^{-1}(V(\mathcal{R}_q)) \subset J_q(T),$$

by considering the *reciprocal image* (or pull-back) over X by id_q of the vertical bundle $V(\mathcal{R}_q)$ of $\mathcal{R}_q$ around its identity solution used by physicists. For this reason, when PDEs are concerned, one may speak about *systems of finite Lie* *equations* for the Lie groupoid $\mathcal{R}_q$ and *systems of infinitesimal Lie equations* for the Lie algebroid R_q. Accordingly, the sheaf of germs of *solutions* of $\mathcal{R}_q$ is a *Lie pseudogroup* of order q, $\Gamma \subset \mathrm{aut}(X)$, where $\mathrm{aut}(X)$ is the pseudogroup of all *local* diffeomorphisms of X preserving the orientation, that is,

$$\begin{aligned}
&\forall f, g \in \Gamma \Rightarrow g \circ f \in \Gamma, \quad \text{whenever } f \text{ and } g \text{ can be composed,}\\
&\forall f \in \Gamma \Rightarrow f^{-1} \in \Gamma,\\
&\mathrm{id}_X \in \Gamma,
\end{aligned}$$

and, with (in general infinite dimensional!) Lie algebra $\Theta \subset T$ the sheaf of germs of solutions of R_q (Θ is a sheaf of Lie algebras). In the sequel Γ will be supposed *transitive*, that is, for every $x, y \in X$ there exists a transformation $f \in \Gamma$ such that $y = f(x)$, and $\mathcal{R}_q$ will be *formally transitive*, i.e., $\pi_0^q : \mathcal{R}_q \to X \times X$ is surjective. Let us give some examples which will be of constant use.

Examples 2.2. (i) On $X = \mathbb{R}$ the Lie pseudogroup Γ of *affine* transformations is given by the solutions of the second order linear ODE

$$\mathcal{R}_2 : y_{xx} = 0, \tag{2.4}$$

and the corresponding linear system is

$$\mathrm{R}_2 : \xi_{xx}(x) = 0. \tag{2.5}$$

(ii) On $X = \mathbb{R}$, if Γ is the Lie pseudogroup of *projective* transformations, it is characterized by the third order non-linear ODE

$$\mathcal{R}_3 : y_{xxx}/y_x - \tfrac{3}{2}\,(y_{xx}/y_x)^2 = 0, \tag{2.6}$$

i.e., the vanishing of the well known Schwarzian derivative, and sections of the corresponding Lie algebroid R_3 are defined by

$$\mathrm{R}_3 : \xi_{xxx}(x) = 0. \tag{2.7}$$

(iii) Similarly, at order one, the unimodular pseudogroup in $X = \mathbb{R}^n$ is defined by the system

$$\mathcal{R}_1 : \partial(y^1, \dots, y^n)/\partial(x^1, \dots, x^n) = 1,$$

while sections of the associated linearized system R_1 are defined [#1] by

$$\mathrm{R}_1 : \xi_r^r(x) = 0.$$

Of course solutions of R_1 are "divergence-free" vector fields.

[#1] It must not be confused with the divergence $\partial_r \xi^r(x)$ of a vector field $\xi(x)$.

Now, if $x_0 \in X$ is a given point, we may introduce the *isotropy Lie groups* $G_q = \mathcal{R}_q(x_0, x_0)$ and $GL_q = \Pi_q(x_0, x_0)$ made by jets with the same source and target $\{x_0\}$, and we have $G_q \subset GL_q$.

The subbundle of Π_q made by jets of transformations with arbitrary source and fixed target $\{x_0\}$ is a *principal bundle* $\Pi_q(X, x_0)$ whose structural group is GL_q; the action is defined by composition at the *target* by jets with the same source and target x_0. Then we may define the bundle $\mathcal{F} = \Pi_q(X, x_0)/G_q$ of homogeneous spaces with typical fiber GL_q/G_q. The image $\mathcal{R}_q(X, x_0)/G_q$ of $\mathcal{R}_q(X, x_0) \subset \Pi_q(X, x_0)$ by the canonical projection $\phi : \Pi_q(X, x_0) \to \mathcal{F}$ makes sense thanks to the Lie groupoid structure of $\mathcal{R}_q$. Moreover, by the formal transitivity of $\mathcal{R}_q$ this image defines a section ω of the bundle $\mathcal{F} \to X$. The dimension of the fibers of $\mathcal{F}$ is equal to $m = \operatorname{codim} \mathcal{R}_q$.

Following Vessiot [24] the groupoid composition at the source commutes with the groupoid composition at the target. Then a groupoid action of $\Pi_q(X, X)$ can also be defined on $\Pi_q(X, x_0)$ by composition at the *source*. This action factors onto a groupoid action of $\Pi_q(X, X)$ on $\mathcal{F}$ by source transformations. Therefore there is on $\mathcal{F}$ an induced structure of a *natural bundle* of order q over X provided by the *association* with $\Pi_q(X, X)$ with an invertible *natural action*

$$\lambda : \mathcal{F} \times_X \Pi_q(X, X) \longrightarrow \mathcal{F}, \qquad \lambda^{-1} : \Pi_q(X, X) \times_X \mathcal{F} \longrightarrow \mathcal{F},$$

defined through the following commutative diagram:

$$\begin{tikzcd} \mathcal{F} \times_X \Pi_q \arrow[r, "\lambda"] \arrow[d] & \mathcal{F} \arrow[d, "\pi"] \\ \Pi_q \arrow[r, "\beta_q"] & X \end{tikzcd}$$

This last description allows us to *lift* [#2] any $f \in \operatorname{aut}(X)$ to a local automorphism of $\mathcal{F}$ fibered over f by defining on the sections of $\mathcal{F}$ an action of *either* jet sections f_q *or* jets of maps $j_q(f)$. The action of f_q [or $j_q(f)$] on a section ω of $\mathcal{F}$ is a new section $\bar{\omega} \equiv f_q(\omega)$ of $\mathcal{F}$ defined by the following commutative diagram:

$$\begin{tikzcd} \mathcal{F} \times_X \Pi_q \arrow[r, "\lambda"] \arrow[d] & \mathcal{F} \arrow[d, "\pi"'] \\ X \arrow[u, shift left, "(\omega, f_q)"] \arrow[r, "f"] & X \arrow[u, shift right, "\bar{\omega} \equiv f_q(\omega)"'] \end{tikzcd} \tag{2.8}$$

[#2] This simple definition ought to be compared to the sophisticated one used for a bundle associated with a principal bundle.

that is to say $\bar{\omega}(f(x)) = \lambda(\omega(x), f_q(x))$. In particular for $\varphi \in \mathrm{aut}(X)$ the *finite transformation laws* $\bar{\omega}(\varphi(x)) = \lambda(\omega(x), j_q(\varphi)(x))$ called *natural transformations* may be used for constructing completely the bundle $\mathcal{F}$ (compare with the tensorial case) by patching together adapted local coordinates for $\mathcal{F}$, (x, u), where $u = (u_1, \ldots, u_m)$ are local coordinates on the fiber of $\mathcal{F}$. We have the formulas

$$\mathcal{F}: \begin{cases} \bar{u} = \lambda(u, j_q(\varphi)(x)) \\ \bar{x} = \varphi(x) \end{cases} . \tag{2.9}$$

Since $\phi(\mathcal{R}_q(X, x_0)) = \omega$ is a section of $\mathcal{F}$, by pulling back ω by a section of Π_q, it is possible to extend ϕ to a surjective morphism $\Phi_\omega : \Pi_q(X, X) \to \mathcal{F}$, $\Phi_\omega(f_q) = f_q^{-1}(\omega)$. Hence one has the *equivalent* definitions found by Vessiot [24].

Definition 2.3 (Vessiot). The Lie groupoid $\mathcal{R}_q \subset \Pi_q$ of order q over X is defined by

$$\mathcal{R}_q = \{f_q \in \Pi_q(X, X) \,/\, f_q(\omega) = \omega\} ; \tag{2.10}$$

the Lie pseudogroup $\Gamma \subset \mathrm{aut}(X)$ is defined by

$$\Gamma = \{f \in \mathrm{aut}(X) \,/\, j_q(f)(\omega) = \omega\}. \tag{2.11}$$

Let us now introduce

Definition 2.4. By a formal differential invariant of order q for Γ we mean a smooth function Φ on Π_q which is invariant under the left action of the defining groupoid $\mathcal{R}_q$, that is,

$$\Phi(g_q \circ f_q) = \Phi(f_q) , \quad f_q \in \Pi_q , \ g_q \in \mathcal{R}_q .$$

Remark 2.5. Recall that a left action is induced by a finite transformation at the *target*. A differential invariant is then a function which stays constant along the orbits of the action map $\Pi_q \times_X \mathcal{R}_q \to \Pi_q$.

Example 2.6. The pull-back $\Phi_\omega(f_q) = f_q^{-1}(\omega)$ is such a differential invariant because

$$\Phi_\omega(g_q \circ f_q) = (g_q \circ f_q)^{-1}(\omega) = (f_q^{-1} \circ g_q^{-1})(\omega) = f_q^{-1}(\omega) = \Phi_\omega(f_q) , \tag{2.12}$$

whenever $g_q \in \mathcal{R}_q$, by eq. (2.10).

Looking at eq. (2.12), and since Φ_ω is invariant under the groupoid action [cf. (2.8)] at the *target*, we may write

$$(f_q^{-1}(g_q^{-1}(\omega)))(x) = \lambda^{-1}((g_q^{-1}(\omega))(x), f_q(x)) = \lambda^{-1}(\omega(y), f_q(x)) ,$$

thanks to the defining equation (2.10) for $g_q \in \mathcal{R}_q$. In other words,

$$\Phi_\omega(y_q) \equiv \lambda^{-1}(\omega(y), y^k_\mu)\,, \quad 1 \le |\mu| \le q\,, \tag{2.13}$$

where the dependence on y of $\Phi_\omega(y_q)$ is only through the section ω. This last formula (2.13) will *define* the differential invariants in local coordinates by means of a finite transformation law (2.8) determined by the pull-back.

By analogy with tensorial bundles, let us give

Definition 2.7. The bundle $\mathcal{F}$ is said to be a bundle of geometric objects of order q, and a given section ω of $\mathcal{F}$ will determine a "structure" on X.

Going to the infinitesimal point of view we may define a *formal Lie derivative* $L(\xi_q)\omega$ of ω with respect to a section ξ_q of $J_q(T)$ and with values in the vector bundle over X,

$$F_0 \;=\; \omega^{-1}(V(\mathcal{F})) \;=\; J_q(T)/\mathrm{R}_q\,,$$

which is the pull-back over X of the vertical bundle of $\mathcal{F}$ by the section ω. It is defined by considering the derivative

$$L(\xi_q)\omega \;=\; \mathrm{d}/\mathrm{d}t|_{t=0}\, f_{q,t}^{-1}(\omega)$$

of a one parameter family of pull-backs of a section $\omega \in \mathcal{F}$. Similarly we also recover the usual *Lie derivative* by taking $f_{q,t} = j_q(f_t)$ $[\xi_q = j_q(\xi)]$,

$$\mathcal{L}(\xi) \;=\; L(j_q(\xi))$$

for any vector field $\xi \in T$. Then we have

Definition 2.8. The Lie algebroid R_q is defined by

$$\mathrm{R}_q \;=\; \{\xi_q \in J_q(T)\,/\,L(\xi_q)\omega = 0\}\,; \tag{2.14}$$

the Lie algebra Θ of the pseudogroup Γ is defined by

$$\Theta \;=\; \{\xi \in T\,/\,\mathcal{L}(\xi)\omega = 0\}. \tag{2.15}$$

Further, we may even introduce the q-th order linear differential operator

$$\mathcal{D}:\; T \longrightarrow F_0,\; \xi \longmapsto \mathcal{L}(\xi)\omega\,,$$

but *the reader must not forget that the section ω may not be a tensor at all!*

Examples 2.9. For the previous examples 2.2 we can indeed perform the following constructions for the natural bundle $\mathcal{F}$.

(i) In the affine case, it turns out that the expression

$$\Phi^{\mathrm{aff}}(y_2) \equiv y_{xx}/y_x \tag{2.16}$$

is a differential invariant of order two for $\mathcal{R}_2$ in the sense of definition 2.4 as is readily checked by acting on the *target* with any section $f_2 \in \mathcal{R}_2$. By performing a transformation at the *source*, $\bar{x} = \varphi(x)$, $\varphi \in \mathrm{aut}(X)$, which commutes with a transformation at the target, the change of differential invariant provides the definition of the bundle of *"affine geometric objects"* (or affine structures) given by the following patching rules for the natural bundle $\mathcal{F}^{\mathrm{aff}}$ [with λ^{-1} in eq. (2.9) instead of λ]:

$$\bar{x} = \varphi(x) \;\Rightarrow\; u \;=\; \bar{u}\,\partial_x\varphi + \partial_x^2\varphi/\partial_x\varphi, \tag{2.17}$$

where ∂_x stands for $\partial/\partial x$ and (x,u) are local adapted coordinates of $\mathcal{F}^{\mathrm{aff}}$.

(ii) Similarly, in the projective case

$$\Phi^{\mathrm{proj}}(y_3) \equiv y_{xxx}/y_x - \tfrac{3}{2}\,(y_{xx}/y_x)^2, \tag{2.18}$$

is a differential invariant of order three for $\mathcal{R}_3$. Following the same line of construction this yields the definition of a *"projective geometric object"* (or projective structure) by the patching rules

$$\bar{x} = \varphi(x) \;\Rightarrow\; v \;=\; \bar{v}\,(\partial_x\varphi)^2 + \{\varphi, x\}, \tag{2.19}$$

where $\{\varphi, x\}$ is the Schwarzian derivative of the local diffeomorphism φ with respect to the coordinate x and where (x,v) are adapted local coordinates on $\mathcal{F}^{\mathrm{proj}}$.

(iii) Likewise, a volume form is characterized by the glueing of a density

$$\bar{x} = \varphi(x) \;\Rightarrow\; g \;=\; \bar{g}\,\partial(\varphi^1,\dots,\varphi^n)/\partial(x^1,\dots,x^n).$$

Now we give

Definition 2.10. More generally one may define an R_q*-connection* χ_q as a splitting of the short exact sequence

$$0 \longrightarrow R_q^0 \longrightarrow \mathrm{R}_q \underset{\pi_0^q}{\overset{\chi_q}{\rightleftarrows}} T \longrightarrow 0, \tag{2.20}$$

provided that the Lie equations are transitive, namely $(\alpha_q, \beta_q) : \mathcal{R}_q \to X \times X$ and $\pi_0^q : \mathrm{R}_q \to T$ are both surjective. Here $\mathrm{R}_q^0 = \mathrm{Ker}\,\pi_0^q$. Equivalently such a connection may also be viewed as a section $\chi_q \in T^* \otimes \mathrm{R}_q$ projecting through π_0^q onto the section $\mathrm{id}_T \in T^* \otimes T$, i.e., $\pi_0^q \circ \chi_q = \chi_0 = \mathrm{id}_T$.

Remark 2.11. For $X = \mathbb{R}$, $n = 1$, the reader will easily check that in the case (2.5) of an affine structure an R_2-connection is just defined by the *same* rule (2.17) of the associated geometrical object; but this is a *pure coincidence*! More precisely under the local transformation $\bar{x} = \varphi(x)$, since $\mathrm{R}_2 \simeq J_1(T)$ thanks to eq. (2.4), we obtain the following local change of jet coordinates:

$$\mathrm{R}_2 : \;\; (\xi, \xi_x, 0) \to (\bar{\xi}, \bar{\xi}_{\bar{x}}, 0),$$

$$\text{with } \bar{\xi} = \partial_x\varphi\,\xi,\; \bar{\xi}_{\bar{x}} = \frac{\partial_x^2\varphi}{\partial_x\varphi}\xi + \xi_x, \tag{2.21a}$$

$$T^* \otimes R_2 : \quad (\xi_{,x}, \xi_{x,x}, 0) \to (\bar{\xi}_{,\bar{x}}, \bar{\xi}_{\bar{x},\bar{x}}, 0),$$
$$\text{with } \bar{\xi}_{,\bar{x}} = \xi_{,x},\ \bar{\xi}_{\bar{x},\bar{x}} = \frac{\partial_x^2 \varphi}{(\partial_x \varphi)^2}\xi_{,x} + \frac{1}{\partial_x \varphi}\xi_{x,x}, \qquad (2.21\text{b})$$

where the second subscript after the comma labels T^*. Since the connection χ_2 does determine a splitting (projecting onto id_T) it can be expressed locally as a section

$$x \mapsto (1, a_{1,x}(x), 0) \equiv (\xi_{,x}(x), -\xi_{x,x}(x), 0).$$

So the last transformation law in (2.21) reads as the definition (2.17) of an affine geometric object for "$u = a_{1,x}$" and therefore χ_2 is *indeed* an *affine connection.*

Similarly, after a slightly tedious but direct computation in the projective case given by eq. (2.7), an R_3-connection χ_3 may be considered as a section of $T^* \otimes R_3 \simeq T^* \otimes J_2(T)$,

$$x \mapsto (1, a_{1,x}(x), a_{2,x}(x), 0) \equiv (\xi_{,x}(x), -\xi_{x,x}(x), -\xi_{xx,x}(x), 0).$$

By projection we can then read off the following connections:

$$x \mapsto (1, a_{1,x}(x), 0)$$

as affine connection, and

$$x \mapsto (1, a_{1,x}(x), a_{2,x}(x))$$

as an $J_2(T)$-connection, for which the components are mixed between themselves by patching. But there exists a combination defining a new R_3-connection whose components do not mix under a transformation

$$x \mapsto (1, a_{1,x}(x), a_{2,x}(x) + \tfrac{1}{2}a_{1,x}^2(x), 0).$$

Again by accident, its third component transforms as in (2.19) for a projective geometric object with "$v = a_{2,x} + \frac{1}{2}a_{1,x}^2$", i.e. as *projective connection.* But it must be of interest to the reader to notice that *two independent* objects are required with their patching law for constructing such a v which is rather a coordinate on a natural bundle.

We shall see below that actually geometrical objects and connections are completely *different* concepts. The former are related to the Janet sequence although they are usually confused with the latter, which arise in the Spencer sequence.

In order to obtain a simple understanding of the *equivalence problem* for structures on X, Vessiot's key idea was to fix the bundle $\mathcal{F}$ but to vary the section. Indeed, if ω and $\bar{\omega}$ are two *given* sections of $\mathcal{F}$ we may look for $f \in \mathrm{aut}(X)$ such that

$$j_q(f)^{-1}(\omega) = \bar{\omega}. \qquad (2.22)$$

The solution of this problem is related to a certain number of *compatibility conditions* usually depending on ω that must be satisfied by $\bar{\omega}$. Once these *formal* *conditions* are fulfilled the *local* solvability of the problem also depends on a difficult analysis due to the existence of counterexamples [20]. We shall only be concerned with the *formal problem*.

In a similar spirit, if we take an arbitrary section ω, the corresponding systems of finite or infinitesimal Lie equations, see definition 2.3, may not be *involutive*. We shall sketch out this situation, referring for the precise definitions to refs. [14,20]. Briefly speaking, the *symbol* $M_q = R_q \cap S_qT^* \otimes T$ of R_q is said to be *involutive* if certain purely algebraic cohomology groups introduced by Spencer [13,14] through the exactness of certain sequences (δ-sequences) vanish. We cannot insist on this delicate and technical notion and the reader must just keep in mind that it can be checked by pure linear (computer) algebra and that it extends the classical definitions given by Janet [16] and Cartan at the beginning of the century. The reader who wishes to have a short survey about symbol and Spencer δ-cohomology is referred to ref. [19].

However, a system of PDEs is said to be *involutive* if it both has an involutive symbol and is *formally integrable*. As this last concept is essential for applications we shall spend some lines on it. Roughly speaking, a system of PDEs is called *formally integrable* if a family of solutions can be constructed as formal power series by r differentiations ($r \to \infty$), in such a way that at each step the new system $\mathcal{R}_{q+r}$ thus obtained and called *r-prolongation* of $\mathcal{R}_q$ brings no new information on the lower order derivatives already computed at a point.

Example 2.12 (Janet). Considering three independent variables (x^1, x^2, x^3) and one unknown y, in the standard notation for derivatives, the system

$$\partial_{33}y - x^2\partial_{11}y = 0, \qquad \partial_{22}y = 0,$$

is not formally integrable because by differentiating twice the first PDE with respect to x^2 and twice the second PDE with respect to x^3 and x^1 leads to $\partial_{112}y = 0$. Such a condition cannot be obtained by differentiating only once each PDE.

Of course, in practice it is *essential* to know whether a system is formally integrable since it is the only way to collect information about the space of solutions without any explicit integration.

Example 2.13. At first sight it is not evident that the space of solutions in the previous example 2.12 is a 12-dimensional vector space over the constants.

Coming back to our pseudogroups, we discover that for an arbitrary ω it is extremely important to know whether $\mathcal{R}_q(\omega)$ is involutive or at least formally integrable. The answer to this question has been sketched by Vessiot in 1903 [24] and the second author has extended it to the general case in refs. [20,21]. The resulting *integrability conditions* for an arbitrary section ω of $\mathcal{F}$ may be written in the form of a non-linear system $\mathcal{B}(c) \subset J_1(\mathcal{F})$ defined by a set of local PDEs,

$$I(j_1(\omega)) = c(\omega), \tag{2.23}$$

where $I : J_1(\mathcal{F}) \to \mathcal{F}_1$ is a certain natural equivariant morphism between natural bundles over $\mathcal{F}$ while $c : \mathcal{F} \to \mathcal{F}_1$ is a natural equivariant section. The main surprise is that *the previous conditions* (2.23) *do not depend on the coordinate system*. Moreover, c only depends *locally* on a finite number of constants called *structure constants* and fulfills algebraic conditions $J(c) = 0$ at most quadratic, which generalize the well known Jacobi identities for those appearing in the Maurer–Cartan equations [20,21].

As a consequence, the equivalence problem can be reformulated by saying that the two sections ω and $\bar{\omega}$ *must* satisfy the same integrability conditions (2.23) *with the same constants*, that is,

$$I(j_1(\omega)) = c(\omega), \qquad I(j_1(\bar{\omega})) = \bar{c}(\bar{\omega}) \quad \text{with } c = \bar{c}. \tag{2.24}$$

Collecting all the preceding results, we obtain the following *non-linear Janet sequence*:

$$\begin{array}{ccccccccc} 0 \longrightarrow \Gamma \longrightarrow \mathrm{aut}(X) & \underset{\omega\circ\alpha}{\overset{\Phi\circ j_q}{\rightrightarrows}} & \mathcal{F} & \underset{c}{\overset{I\circ j_1}{\rightrightarrows}} & \mathcal{F}_1, \\ f & \longmapsto & j_q(f)^{-1}(\omega), & & \end{array} \tag{2.25}$$

where Φ is the same as in example 2.6. The lower arrow in the first set of double arrrows means that the kernel of Φ with respect to a section ω of $\mathcal{F}$, $\mathrm{Ker}_\omega(\Phi)$, is simply Γ and the second set of double arrows expresses eq. (2.23). For the sake of completeness we indicate in the following diagram how to consider the equivalence problem:

$$\begin{tikzcd}
\mathcal{F}\times_X \Pi_q \arrow[rr, "\lambda"] \arrow[d, shift left] \arrow[drr, "\beta_q"] & & \mathcal{F} \arrow[d, shift left] \\
X \arrow[u, shift left, "\bar{\omega}"] \arrow[u, bend right, "j_q(f)"'] \arrow[rr, "f"] & & X \arrow[u, shift left, "\omega"']
\end{tikzcd} \tag{2.26}$$

with $\lambda(j_q(f)^{-1}(\omega)(x), \partial_\mu f^k(x)) = \omega(f(x))$.

Finally by linearization we obtain the *linear Janet sequence*, where each differential operator represents the *compatibility conditions* for the preceding one,

$$0 \longrightarrow \Theta \longrightarrow T \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \cdots \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0. \tag{2.27}$$

We now present a few relevant examples

Examples 2.14. (i) $\omega \in \mathcal{F} = S_2T^*$, $\det\omega \neq 0$. In this case the well known integrability condition is constant Riemannian curvature as found by Eisenhart.

(ii) $\omega \in \mathcal{F} = T^* \otimes \mathcal{G}$, where $\mathcal{G}$ is the Lie algebra of some Lie group G acting simply and transitively on X (or even itself). In that case the integrability conditions are the well known Maurer–Cartan equations.

(iii) $\omega \in \mathcal{F} = T^* \otimes T$, $\omega^2 = -\mathrm{id}_T$ with $n = 2p$. In that case of almost complex analytic structures, the integrability conditions are the so called Nijenhuis conditions.

(iv) $\omega = (\alpha, \beta) \in \mathcal{F} = T^* \times_X \wedge^2 T^*$ with $n = 2$. Thus the integrability condition is $\mathrm{d}\alpha = c\beta$. Consequently by using (2.24), $\omega = (x^2 \mathrm{d}x^1, \mathrm{d}x^1 \wedge \mathrm{d}x^2)$ and $\bar{\omega} = (\mathrm{d}x^1, \mathrm{d}x^1 \wedge \mathrm{d}x^2)$ are *not* equivalent because $c = -1$ and $\bar{c} = 0$, i.e., [cf. (2.22)] asking for $f \in \mathrm{aut}(X)$ such that $\bar{\omega} = j_1(f)^{-1}(\omega)$ has no solution.

2.2. SPENCER SEQUENCES

Using definitions and objects previously introduced we shall now construct the *first non-linear Spencer sequence*

$$0 \longrightarrow \mathrm{aut}(X) \xrightarrow{j_{q+1}} \Pi_{q+1}(X,X) \xrightarrow{\overline{D}} T^* \otimes J_q(T) \xrightarrow{\overline{D}'} \wedge^2 T^* \otimes J_{q-1}(T)\,. \quad (2.28)$$

As the involved bundles have already been defined, it just remains to construct the above operators. First recall that the operator of order $q+1$,

$$j_{q+1} : f^k(x) \longmapsto \{\partial_\mu f^k(x) \,/\, 0 \le |\mu| \le q+1\}, \quad k = 1, \ldots, n,$$

sends $f \in \mathrm{aut}(X)$ to $j_{q+1}(f) \in \Pi_{q+1}$. The algebraic structure of $J_1(\Pi_q)$ is that of a *monoid* with the composition defined by

$$j_1(g_q) \circ j_1(f_q) \;=\; j_1(g_q \circ f_q)$$

whenever $g_q \circ f_q$ is defined for f_q, $g_q \in \Pi_q$. The inclusion $\Pi_{q+1} \subset J_1(\Pi_q)$ together with the groupoid structure of Π_{q+1} implies that the section $f_{q+1}^{-1} \circ j_1(f_q)$ is a well defined section of $J_1(\Pi_q)$ over the section $f_q^{-1} \circ f_q = \mathrm{id}_q$ of Π_q. The fact that id_{q+1} is also a section of $\Pi_{q+1} \subset J_1(\Pi_q)$ over the same section id_q of Π_q combined with the property that $J_1(\Pi_q)$ is an *affine* bundle over Π_q modelled on $T^* \otimes V(\Pi_q)$ leads us to construct the operator $\overline{D}$ as

$$\chi_q \;=\; \overline{D} f_{q+1} \;=\; f_{q+1}^{-1} \circ j_1(f_q) - \mathrm{id}_{q+1}, \quad (2.29)$$

where χ_q is a section of the reciprocal image of $T^* \otimes V(\Pi_q)$ by id_q^{-1} (induced bundle), that is, a section of $T^* \otimes J_q(T)$. By definition of the operator $\overline{D}$ we obtain

$$\begin{aligned} \overline{D}(g_{q+1} \circ f_{q+1}) &= f_{q+1}^{-1} \circ g_{q+1}^{-1} \circ j_1(g_q) \circ j_1(f_q) - \mathrm{id}_{q+1} \\ &= f_{q+1}^{-1} \circ \overline{D} g_{q+1} \circ j_1(f_q) + \overline{D} f_{q+1}, \end{aligned} \quad (2.30)$$

which can be recognized as a *gauge transformation* induced by a *source* transformation.

Let us now introduce the key tool which is the *Spencer operator*.

Definition 2.15. The (linear) Spencer operator is a first order differential operator defined by

$$\begin{aligned} D:\ J_{q+1}(T) &\longrightarrow T^* \otimes J_q(T), \\ \xi_{q+1} &\longmapsto j_1(\xi_q) - \xi_{q+1}, \end{aligned} \tag{2.31}$$

with

$$(D\xi_{q+1})^k_{\mu,i} = \partial_i \xi^k_\mu - \xi^k_{\mu+1_i}. \tag{2.32}$$

This operator can also be defined in a non-linear way on $J_{q+1}(X,X)$ in order to measure the difference between the two sections f_{q+1} and $j_1(f_q)$ over f [22, p. 134]. We have $D \circ j_q \equiv 0$.

We may extend the Spencer operator to an operator

$$D : \wedge^s T^* \otimes J_{q+1}(T) \longrightarrow \wedge^{s+1} T^* \otimes J_q(T)$$

by setting

$$\begin{aligned} D(\alpha \otimes \xi_{q+1}) &\equiv \mathrm{d}\alpha \otimes \xi_q + (-1)^s \alpha \wedge D\xi_{q+1}, \\ &\forall \alpha \in \wedge^s T^*, \quad \forall \xi_{q+1} \in J_{q+1}(T), \end{aligned} \tag{2.33}$$

where d is the differential on forms.

By linearity it is possible to construct the following *algebraic bracket*:

$$\{J_{q+1}(T), J_{q+1}(T)\} \subset J_q(T) \tag{2.34}$$

from the usual bracket of vector fields through the relation

$$\{j_{q+1}(\xi), j_{q+1}(\eta)\} \stackrel{\text{def}}{=} j_q([\xi,\eta]), \quad \forall \xi,\eta \in T \tag{2.35}$$

(given by the Leibniz rule) by replacing the derivatives $\partial_\mu \xi$ and $\partial_\lambda \eta$ by the jet coordinates ξ_μ and η_λ, respectively, see formula (C.1) in appendix C. Note that due to the change of the jet order in (2.34) this bracket is in general *not* a Lie bracket.

Next for any $\chi_q \in T^* \otimes J_q(T)$, by linearity we can define the operator $\overline{D}'$ by setting

$$(\overline{D}'\chi_q)(\xi,\eta) = (D\chi_q)(\xi,\eta) - \{\chi_q(\xi), \chi_q(\eta)\}, \ \forall \xi,\eta \in T, \tag{2.36}$$

with the following important property:

$$D^2 = 0 \Rightarrow \overline{D}' \circ \overline{D} = 0.$$

Notice that expression (2.36) belongs to $J_{q-1}(T)$.

Finally an inductive argument proves that the general first non-linear Spencer sequence (2.28) is *locally exact* and admits the restriction

$$0 \longrightarrow \Gamma \xrightarrow{j_{q+1}} \mathcal{R}_{q+1} \xrightarrow{\overline{D}} T^* \otimes \mathrm{R}_q \xrightarrow{\overline{D}'} \wedge^2 T^* \otimes \mathrm{R}_{q-1}, \tag{2.37}$$

which is not necessarily locally exact at $T^* \otimes \mathrm{R}_q$. Delicate explicit computations prove that neither $\overline{D}$ nor $\overline{D}'$ are formally integrable. Overcoming this problem leads to the *second non-linear Spencer sequence* exhibiting better formal properties. For details, see refs. [21,22].

Also, the gauge transformations (2.30) *with respect to* $\mathcal{R}_{q+1}$ *restrict to* $T^* \otimes R_q$ and exchange the solutions of $\overline{D}'$, i.e., act on the sections of the kernel of $\overline{D}'$. We define

Definition 2.16. The gauge transformation of a section $\chi_q \in T^* \otimes R_q$ (which is not necessarily a connection) is the transformation

$$\chi_q \longmapsto f_{q+1}^{-1} \circ \chi_q \circ j_1(f_q) + \overline{D} f_{q+1} \tag{2.38}$$

for any (invertible) section f_{q+1} of $\mathcal{R}_{q+1}$.

This definition shows the link with gauge theory and proves that new operations might be performed in the framework of groupoids which have *no equivalent* in the use of principal bundles.

Moreover, a direct computation of $\overline{D}'$ in *local coordinates* for $q = 2$ gives torsion and curvature, see eq. (2.42) below.

According to ref. [22, p. 233], we can express a section $\chi_q = \overline{D} f_{q+1} \in \operatorname{Ker} \overline{D}' \subset T^* \otimes R_q$ locally by $\chi_q(x) = (\chi^k_{\mu,i}(x))_{0 \leq |\mu| \leq q}$. Let us take $q = 1$, and since $\det(f_i^k(x)) \neq 0$ define the inverse by

$$g_k^i(x) f_j^k(x) = \delta_j^i. \tag{2.39}$$

For a section $f_2 : x \mapsto (x, f^k(x), f_i^k(x), f_{ij}^k(x))$ of $\mathcal{R}_2$, we successively get for $\overline{D}$ [see eq. (2.29)]

$$\begin{aligned} \chi_1 &= \overline{D} f_2 \in T^* \otimes R_1, \\ \chi^k_{,j} &= g_l^k \partial_j f^l - \delta_j^k \equiv A_j^k - \delta_j^k, && (2.40\text{a}) \\ \chi^k_{i,j} &= g_l^k (\partial_j f_i^l - A_j^r f_{ri}^l), && (2.40\text{b}) \end{aligned}$$

with the mixed object made from one-jets and first derivatives,

$$A_j^k(x) = g_r^k(x) \partial_j f^r(x) , \tag{2.41}$$

and for $\overline{D}$, the compatibility conditions $\overline{D}'$, with $q = 2$ [see eq. (2.36)], write

$$\begin{aligned} &\partial_i \chi^k_{,j} - \partial_j \chi^k_{,i} - \chi^k_{i,j} + \chi^k_{j,i} - (\chi^r_{,i} \chi^k_{r,j} - \chi^r_{,j} \chi^k_{r,i}) = 0, && (2.42\text{a}) \\ &\underbrace{\underline{\partial_i \chi^k_{l,j} - \partial_j \chi^k_{l,i} - \chi^k_{li,j} +} \chi^k_{lj,i}}_{D\chi_2} \\ &\quad - (\underbrace{\chi^r_{,i} \chi^k_{lr,j} + \chi^r_{l,i} \chi^k_{r,j} - \chi^r_{l,j} \chi^k_{r,i} - \chi^r_{,j} \chi^k_{lr,i}}_{\{\chi_2, \chi_2\}}) = 0, && (2.42\text{b}) \end{aligned}$$

and so on. Equations ((2.42)a,b) define a section of $\wedge^2 T^* \otimes R_1$, while eq. (2.42a) alone defines a section of $\wedge^2 T^* \otimes T$, in the projective limit of the various Spencer sequences (2.37). Clearly in the covariant expression $\overline{D}'\chi_q$ by construction both the D-term and the (quadratic) terms of the algebraic bracket are *separately* covariant.

Remark 2.17. Notice that, if χ_q is (minus) a *connection*, cf. definition 2.10, i.e., $\pi_0^q \circ \chi_q = \chi_0 = -\mathrm{id}_T$, then (2.40a) implies that the matrix A vanishes, $A \equiv 0$. By eqs. (2.39) and (2.41) the projection $\pi_0^q \circ f_q = f$ is a *constant* map section of $X \times X$ inducing a section f_q of the *principal bundle* $\Pi_q(X, x_0)$ for which *no* prolongation can be defined since the target is fixed.

Writing $\overline{D}'\chi_q = \tau_{q-1} \in \wedge^2 T^* \otimes J_{q-1}(T)$ for an *arbitrary* section $\chi_q \in T^* \otimes J_q(T)$, we obtain as compatibility conditions for $\overline{D}'$ the following *Bianchi identities* [22, p. 236]:

$$D\tau_{q-1}(\xi,\eta,\zeta) + \overset{\hookrightarrow}{(\xi,\eta,\zeta)}\ \{\tau_{q-1}(\xi,\eta), \chi_{q-1}(\zeta)\} = 0, \tag{2.43}$$

which are the evaluation of an element of $\wedge^3 T^* \otimes J_{q-2}(T)$ on vector fields and where $\overset{\hookrightarrow}{(...)}$ stands for the summation over the cyclic permutations of its arguments. In this intrinsic presentation *no* τ_q is involved in $D\tau_{q-1}$.

The corresponding infinitesimal version of the gauge transformation (2.38) is obtained by defining an action of R_{q+1} on $T^* \otimes \mathrm{R}_q$. This gives

Definition 2.18. For a section $\xi_{q+1} \in \mathrm{R}_{q+1}$, the infinitesimal gauge transformation of a section $\chi_q \in T^* \otimes \mathrm{R}_q$ reads

$$\delta\chi_q = \delta_{\xi_{q+1}}\chi_q = L(j_1(\xi_{q+1}))\chi_q + D\xi_{q+1}, \tag{2.44}$$

where the action of the formal Lie derivative $L(j_1(\xi_{q+1}))$ on $T^* \otimes \mathrm{R}_q \subset T^* \otimes J_q(T)$ is split into an action of the formal Lie derivative $L(\xi_{q+1})$ on R_q and an action of the ordinary Lie derivative $\mathcal{L}(\xi) = L(j_1(\xi))$ on T^*, i.e., with $i(\cdot)$ denoting the interior product,

$$i(\zeta)L(j_1(\xi_{q+1}))\chi_q = L(\xi_{q+1})(i(\zeta)\chi_q) - \chi_q([\xi,\zeta]) . \tag{2.45}$$

Moreover, there exists an isomorphism of R_{q+1} defined through

$$\xi'_{q+1} = \xi_{q+1} + \chi_{q+1}(\xi) \tag{2.46}$$

if and only if χ_{q+1} is chosen to be projectable onto χ_q, i.e., $\chi_q = \pi_q^{q+1} \circ \chi_{q+1}$ with $\det A \neq 0$. Thus the infinitesimal gauge transformation (2.44) is written as

$$\delta\chi_q = \delta_{\xi'_{q+1}}\chi_q = D\xi'_{q+1} - \{\chi_{q+1}(\cdot), \xi'_{q+1}\}, \tag{2.47}$$

where the algebraic bracket (2.34) on $J_{q+1}(T)$ with values in $J_q(T)$ occurs. It is tempting to see this last formula (2.47) as a generalization of the well known "infinitesimal gauge transformations" for a Lie group G, see, e.g., refs. [1,25],

$$\delta_\xi a = \mathrm{d}\xi + [a, \xi] , \tag{2.48}$$

where a is the usual "Yang–Mills potential" $a \in T^* \otimes \mathcal{G}$ and ξ is a map $X \to \mathcal{G}$ where $\mathcal{G} = \mathrm{Lie}\, G$, but as quoted above the algebraic bracket is no longer a Lie bracket.

We now give the explicit expressions for the variations (2.44) and (2.47) of $\chi_1 \in T^* \otimes \mathrm{R}_1$ [22, p. 272] [or see formula (A.1) in appendix A],

$$\delta\chi^k_{,i} = (\partial_i\xi^k - \xi^k_i) + (\xi^r\partial_r\chi^k_{,i} + \chi^k_{,r}\partial_i\xi^r - \chi^r_{,i}\xi^k_r) \tag{2.49a}$$

$$= (\partial_i\xi'^k - \xi'^k_i) + (\chi^k_{r,i}\xi'^r - \chi^r_{,i}\xi'^k_r), \tag{2.49b}$$

$$\delta\chi^k_{j,i} = (\partial_i\xi^k_j - \xi^k_{ij}) + (\xi^r\partial_r\chi^k_{j,i} + \chi^k_{j,r}\partial_i\xi^r + \chi^k_{r,i}\xi^r_j - \chi^r_{j,i}\xi^k_r - \chi^r_{,i}\xi^k_{jr}) \tag{2.49c}$$

$$= (\partial_i\xi'^k_j - \xi'^k_{ij}) + (\chi^k_{rj,i}\xi'^r + \chi^k_{r,i}\xi'^r_j - \chi^r_{j,i}\xi'^k_r - \chi^r_{,i}\xi'^k_{jr}). \tag{2.49d}$$

In the particular case of complex analytic transformations (Cauchy–Riemann), the projection $\pi^{q+1}_0 \circ \xi_{q+1} = \xi$ at order zero of a section of the Lie groupoid R_{q+1} is here an *arbitrary vector field*, according to the following picture:

$$\begin{array}{rcl} \underline{\text{Lie Group}} & - - - - \to & \underline{\text{Lie Groupoid}} \\ \text{gauging of the group } G & - - - - \to & \text{section of the groupoid } \mathcal{R}_q \\ \text{gauging of the algebra } \mathcal{G} & - - - - \to & \text{section of the algebroid } \mathrm{R}_q \end{array}$$

Applying twice the transformation (2.44) and antisymmetrizing with respect to two sections $\xi_{q+1}, \eta_{q+1} \in \mathrm{R}_{q+1}$ we obtain the following

Theorem 2.19. *The generalization to Lie groupoids of the known results for Lie groups is*

$$[\delta_{\xi_{q+1}}, \delta_{\eta_{q+1}}]\chi_q = \delta_{[\eta_{q+1},\xi_{q+1}]}\chi_q, \tag{2.50}$$

where [,] *is a differential bracket defined on the sections of* $J_q(T)$ *by*

$$[\eta_q, \xi_q] = \{\eta_{q+1}, \xi_{q+1}\} + i(\eta)D\xi_{q+1} - i(\xi)D\eta_{q+1}. \tag{2.51}$$

Proof. See appendix A.

The differential bracket (2.51) is a Lie algebra bracket and it is well defined thanks to the compensating terms depending on the Spencer operator D [21,22] (it no longer depends on the lifts of order $q+1$) and projects onto the ordinary bracket on T. Moreover, we have the following characterization [22] for a system of infinitesimal Lie equations $[\mathrm{R}_q, \mathrm{R}_q] \subset \mathrm{R}_q$. Hence this theorem simply reflects the Lie algebroid structure of R_q.

Notice that, since the second formulation (2.47) depends on the lift χ_{q+1} of χ_q, it should not have been possible to use it in the above manipulations, although it is similar to (2.48) for Lie groups. However, a direct computation gives

$$[\delta_{\xi'_{q+1}}, \delta_{\eta'_{q+1}}]\chi_q = \delta_{\{\eta'_{q+1},\xi'_{q+1}\}}\chi_q, \tag{2.52}$$

Table 1
Lie groups and generalizing Lie groupoids

Concept	Lie group	Lie groupoid
Composition	$G \times G \to G$	$\mathcal{R}_q \times_X \mathcal{R}_q \to \mathcal{R}_q$
Inversion	$G \to G$	$\mathcal{R}_q \to \mathcal{R}_q$
Identity	$e \in G$	$\mathrm{id}_q \in \mathcal{R}_q$
Lie algebra	$\mathcal{G} = T_e G$	$\mathrm{R}_q = \mathrm{id}_q^{-1}(V(\mathcal{R}_q)) \subset J_q(T)$
Bracket	$[\mathcal{G}, \mathcal{G}] \subset \mathcal{G}$	$[\mathrm{R}_q, \mathrm{R}_q] \subset \mathrm{R}_q$

with the relation

$$\{\xi'_{q+1}, \eta'_{q+1}\} = [\xi_q, \eta_q] + \chi_q([\xi, \eta]) + i(\eta)\delta_{\xi_{q+1}}\chi_q - i(\xi)\delta_{\eta_{q+1}}\chi_q. \quad (2.53)$$

We can summarize the construction in table 1 by saying that Lie groupoids seem to provide an appropriate generalization of Lie groups.

2.3. THE LINK BETWEEN THE NON-LINEAR JANET AND SPENCER SEQUENCES

We arrive at the most delicate point of this first part, namely to establish a link of a cohomological nature between the non-linear Janet sequence and the first non-linear Spencer sequence. We will sketch the main pertinent arguments making this result possible ; for a complete proof see refs. [21,22].

Let $\bar{\omega}$ be a section of $\mathcal{F}$ satisfying the same integrability conditions as ω, i.e. (2.24). We may always find a section $f_q \in \Pi_q$ such that $f_q^{-1}(\omega) = \bar{\omega}$ because Π_q acts transitively in the groupoid sense on $\mathcal{F}$, which is a quotient of Π_q. Since $\mathcal{F}$ is a natural bundle of order q, $J_1(\mathcal{F})$ is a natural bundle of order $q + 1$. Therefore we may find a section $f_{q+1} \in \Pi_{q+1}$ such that

$$f_{q+1}^{-1}(j_1(\omega)) = j_1(\bar{\omega}) = j_1(f_q^{-1}(\omega)) = j_1(f_q)^{-1}(j_1(\omega)).$$

Accordingly

$$j_1(\omega) = f_{q+1}(j_1(\bar{\omega})) = j_1(f_q)(j_1(\bar{\omega})),$$

and therefore

$$(f_{q+1}^{-1} \circ j_1(f_q))^{-1}(j_1(\bar{\omega})) = j_1(\bar{\omega}).$$

One can prove [22] that this last relation implies that

$$\chi_q = \overline{D} f_{q+1} \in T^* \otimes \mathrm{R}_q(\bar{\omega}),$$

which therefore belongs to $\mathrm{Ker}\,\overline{D}'$, i.e., $\overline{D}'\chi_q = 0$.

Thus we have related cocycles at $\mathcal{F}$ in the Janet sequence [i.e., $I \circ j_1(\omega) = c(\omega)$, $I \circ j_1(\bar{\omega}) = c(\bar{\omega})$ for the *same* c (!)] with cocycles at $T^* \otimes \mathrm{R}_q$ in the Spencer sequence *relative* to $\bar{\omega}$ the roles of source and target are reversed through this correspondence. This fact is not evident at all and will be important for applications.

Finally we can remark that, if $f_{q+1}, f'_{q+1} \in \Pi_{q+1}$ are such that $f^{-1}_{q+1}(j_1(\omega)) = f'^{-1}_{q+1}(j_1(\omega)) = j_1(\bar{\omega})$, we deduce the equality $(f^{-1}_{q+1} \circ f'_{q+1})(j_1(\bar{\omega})) = j_1(\bar{\omega})$. Therefore one may find a section $g_{q+1} \in \mathcal{R}_{q+1}(\bar{\omega})$ such that $f'_{q+1} = f_{q+1} \circ g_{q+1}$. The new section $\chi'_q = \bar{D} f'_{q+1}$ thus obtained differs from the previous χ_q by a gauge transformation (2.38) [22]. We can summarize this result in the following main

Theorem 2.20. *Gauge transformations at $T^* \otimes R_q$ in the first non-linear Spencer sequence correspond to natural transformations at $\mathcal{F}$ in the non-linear Janet sequence.*

Remark 2.21. There is a fundamental difference between these two differential sequences. The *bigger* the pseudogroup Γ, the *smaller* the number of differential invariants in the non-linear Janet sequence, but, the *greater* the dimension of the bundles in the first non-linear Spencer sequence. This fact suggests the specific role of the latter sequence in mathematical physics.

Remark 2.22. In the *particular* situation of order *one*, $q = 1$ (which will be met in section 3) the differential invariants are functions of $j_1(f)$, $f \in \mathrm{aut}(X)$, and thus [#3] of $j_1(f_1)$, which are invariant under the action of the pseudogroup Γ at the *target*. However, the images by $\bar{D}$ of two sections $f_{q+1}, f'_{q+1} \in \mathcal{R}_{q+1}$ are equal if and only if $f'_{q+1} = j_{q+1}(g) \circ f_{q+1}$ whenever $g \in \Gamma$. It follows that the differential invariants of order one are functions of the images χ_0 of the first Spencer operator $\bar{D}$.

Accordingly, finite (infinitesimal) gauge transformations of χ_0 induce finite (infinitesimal, which is the Lie derivative) natural transformations of the geometric object associated to the differential invariant. In particular, at the infinitesimal level, it should be noticed that the differential bracket for Lie algebroids generalizes the bracket for Lie algebras and projects onto the ordinary bracket for vector fields at order zero. For a classical example in the framework of Riemannian geometry and well known in continuum mechanics of micropolar (Cosserat) media, see ref. [22, p. 294].

3. The non-linear Janet sequence for the 2-d conformal pseudogroup

The purpose of this section is to construct various non-linear Janet sequences for the pseudogroup Γ of complex analytic transformations of a 2-d manifold X. We shall go into some details even though a few steps could be considered

[#3] This is due to the canonical projection $J_1(J_1(X,X)) \to J_1(X,X)$.

trivial. But we prefer to insist on some points of the construction hoping to help the reader catch the philosophy.

We first consider the Cauchy–Riemann linear system for complex analytic transformations. In (real) jet coordinates it is written as

$$\mathcal{R}_1 : \begin{cases} y_1^1 - y_2^2 = 0 \\ y_2^1 + y_1^2 = 0 \end{cases} , \tag{3.1}$$

and we can notice the following very particular property of the one-jet composition that will be used later on [22, p. 185], namely that the multiplication law of Jacobian matrices is commutative:

$$\begin{aligned} \begin{pmatrix} M & -N \\ N & M \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} &= \begin{pmatrix} MA - NB & -(NA + MB) \\ NA + MB & MA - NB \end{pmatrix} \\ &= \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} M & -N \\ N & M \end{pmatrix} , \end{aligned} \tag{3.2}$$

where A, B, M, N are real numbers. The real dimension of fibers of $\mathcal{R}_1$ is then $2(\text{zero-jets}) + 2(\text{one-jets}) = 4$ and the codimension of the Lie algebroid $\mathrm{R}_1 \subset J_1(T)$ is $m = 2$. So there will be two real differential invariants characterizing $\mathcal{R}_1$.

In order to adapt the notation with the common one used in refs. [3–5,10] we shall introduce source coordinates $(z, \bar{z})$ $(z = x^1 + \mathrm{i}x^2)$, where $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$ will be the derivatives with respect to these coordinates, and complex target coordinates $(Z, \overline{Z})$, $(Z = y^1 + \mathrm{i}y^2)$, with $n = 2$. So Γ is turned into the complex pseudogroup defined by the well known Cauchy–Riemann system of PDEs for holomorphic transformations $\bar{\partial} Z = 0$. The system $\mathcal{R}_1 \subset \Pi_1(X, X)$ is obtained from the Cauchy–Riemann PDEs by substituting jet coordinates

$$\mathcal{R}_1 : Z_{\bar{z}} = 0 . \tag{3.3}$$

Now, in this complex formulation the only parametric derivative obtained by prolongation at any order q is just $\partial^q Z$ since

$$\mathcal{R}_q : Z_{\alpha + 1_{\bar{z}}} = 0 , \quad 0 \le |\alpha| = j + \bar{j} \le q - 1 , \tag{3.4}$$

in a multi-index notation. Thus, all the symbols $M_q = \mathrm{R}_q \cap S_q T^* \otimes T$ have real dimension 2 and are involutive. It follows that the system $\mathcal{R}_1$ is *involutive* and the various projections $\mathcal{R}_{q+r} \to \mathcal{R}_q$ define a *flag* configuration. It is in order to limit the infinite number $2+2+2+\cdots$ of arbitrary jets that the authors of ref. [10] tried to introduce subpseudogroups of Γ of *finite type*, those having an even finite number of jets or equivalently having a null symbol at a certain order.

Let us concentrate on the geometric objects associated to the complex analytic transformations. It is well known that the sections of the natural bundle $\mathcal{F}$ can be identified with mixed tensors $J \in T^* \otimes T$ such that $J^2 = -\mathrm{id}_T$. When $n = 2$ the parametrization of a *general* complex structure J can be carried out in a one-to-one way by using the Beltrami parametrization of complex structures. More precisely there is a unique $(-1,1)$-differential [#4] $\mu_{\bar z}^{\,z} \equiv \mu$ with $|\mu| < 1$ associated to a given J through (see, e.g., ref. [26])

$$J_\mu = \frac{\mathrm{i}}{1-\mu\bar\mu}\begin{pmatrix} 1+\mu\bar\mu & 2\mu \\ -2\bar\mu & -(1+\mu\bar\mu)\end{pmatrix} = \begin{pmatrix} 1 & \mu \\ \bar\mu & 1\end{pmatrix}^{-1} J_0 \begin{pmatrix} 1 & \mu \\ \bar\mu & 1\end{pmatrix}, \tag{3.5}$$

where $J_0 = \left(\begin{smallmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i}\end{smallmatrix}\right)$ in complex coordinates. The "complex number" μ is not constrained by any compatibility condition. The natural bundle $\mathcal{F}$ in this complex formulation can be identified with a smooth bundle over the surface X with respect to its C^∞-structure and a section μ of $\mathcal{F}$ is the (complex) geometrical object parametrizing the complex structures, we refer the reader to ref. [27] for this point. The *special* section giving rise to Γ, the sheaf of solutions of $\mathcal{R}_1$, is $\mu = 0$, that is the above J_0 or $J_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$ in real coordinates. Therefore X is endowed with the complex analytic structure given by $\mu = 0$.

Now, the complex differential invariant of order one for $\mathcal{R}_1$ is the following expression in jet coordinates and therefore gives the differential invariant for the holomorphic transformations:

$$\Phi(f_1(z,\bar z)) \equiv \frac{Z_{\bar z}}{Z_z} \;\Rightarrow\; \Phi(j_1(f)(z,\bar z)) \equiv \frac{\bar\partial Z}{\partial Z} \quad \text{for } f \in \Gamma, \tag{3.6}$$

as is readily checked by acting with $\mathcal{R}_1$ at the *target* (Z is varied while z is kept fixed).

The transition laws of the natural bundle $\mathcal{F}$ are obtained by acting at the *source* with $\varphi \in \mathrm{aut}(X)$, $(z,\bar z) \overset{\varphi}{\longmapsto} (w,\bar w)$; first we get the change of differential invariants

$$\frac{Z_{\bar w}}{Z_w} = \lambda(\Phi, j_1(\varphi)(z,\bar z)) = \frac{\Phi\partial w - \bar\partial w}{\bar\partial \bar w - \Phi\partial\bar w},$$

and if $(z,\bar z;\mu)$ and $(w,\bar w;\nu)$ are local coordinates for $\mathcal{F}$, then the pull-back λ^{-1} gives

$$\mathcal{F}:\; (w,\bar w) = \varphi(z,\bar z) \Rightarrow \mu(z,\bar z) = \frac{\bar\partial w + \nu\circ\varphi(z,\bar z)\bar\partial\bar w}{\partial w + \nu\circ\varphi(z,\bar z)\partial\bar w}. \tag{3.7}$$

In particular, if $\varphi \in \Gamma$, that is, a holomorphic change of coordinates on X, then μ behaves like a $(-1,1)$-differential (tensor!).

According to example 2.6 and eq. (2.13), for the given "null section" (abuse of language) it is readily seen that Φ is indeed its pull-back by $f_1 \in \Pi_1$,

[#4] Warning(!): This symbol μ used from now on has nothing to do with a multi-index notation as introduced in section 2.

$\Phi(f_1(z,\bar{z})) \equiv \lambda^{-1}(0, f_1(z,\bar{z}))$. Using Vessiot's definition 2.3, $f_1 \in \mathcal{R}_1$ if and only if it preserves the null section. Since Φ is a $\mathcal{R}_1$-differential invariant, $\mathcal{R}_1$ can equivalently be defined by its *Lie form*, the section $\mu \equiv 0$ being the evaluation of Φ at the one-jet of the identity map

$$\mathcal{R}_1 : \ \Phi(f_1(z,\bar{z})) = \Phi(\mathrm{id}_1(z,\bar{z})) \iff Z_{\bar{z}}/Z_z = 0 . \tag{3.8}$$

As another application of section 2, let us now consider the equivalence problem. Giving two sections of $\mathcal{F}$, the special complex structure $\mu \equiv 0$ on the *target* and the *general* (arbitrary) complex structure μ on the *source*, we consider their differential link (if it exists) through the pull-back, cf. eq. (2.22),

$$\mu(z,\bar{z}) = (j_1(\varphi)^{-1}(0))(z,\bar{z}) = \lambda^{-1}(0, j_1(\varphi)(z,\bar{z})),$$

which is defined in complex coordinates by the system of one complex (two real) PDEs

$$\overline{\partial} Z/\partial Z = \mu(z,\bar{z}), \tag{3.9}$$

that is, we ask for a local diffeomorphism expressed in target local coordinates by $(Z, \overline{Z})$ such that $j_1(Z,\overline{Z})^{-1}(0) = \mu$.

The analogy with the 2-d conformal pseudogroup comes from an analogy between the real defining PDEs (3.1).

Remark 3.1. Let us add that eq. (3.5) is related to the *equivalent* formulation of the equivalence problem but in terms of the tensor J. It reads

$$(J_0)^k_l(y)\frac{\partial y^l}{\partial x^j}\frac{\partial x^i}{\partial y^k} = (J_\mu)^i_j(x)$$

with the corresponding differential invariant

$$\Phi^i_j \equiv (J_0)^k_l(y)\frac{\partial y^l}{\partial x^j}\frac{\partial x^i}{\partial y^k} ,$$

and $\mathcal{R}_1$ will be equivalently defined in the Lie form

$$\mathcal{R}_1 : \ (J_0)^k_l(f(x))f^l_j(x)g^i_k(x) = (J_0)^i_j(x) ,$$

where $g^j_r(x)f^r_i(x) = \delta^j_i$, see eq. (2.39).

Any subpseudogroup $\Gamma' \subset \Gamma$ of Γ will be defined by a subsystem and of course it will have in general more differential invariants. Hereafter, as examples, we shall restrict ourselves to the affine and projective subpseudogroups of the complex analytic transformations while we shall find again by another technique the corresponding results stated in ref. [10]. The general procedure consists in looking for Lie subgroups of greater and greater dimension in order to describe by a progressive saturation procedure the jets of $\mathcal{R}_q$ when $q \to \infty$. The method outlined here shows that one can avoid the use of the "linear connection" built in ref. [10]. The question concerning the necessity of extending the construction of this "connection" at higher order can be thus addressed.

The affine case. Translating the results of section 1 into complex coordinates let us characterize the affine complex transformations by its differential invariants [see example 2.9(i)] we have to solve the second order non-linear system of PDEs

$$\Phi \equiv \bar{\partial} Z/\partial Z = \mu , \qquad \Phi^{\mathrm{aff}} \equiv \partial^2 Z/\partial Z = u . \tag{3.10}$$

We shall use two methods for finding the compatibility conditions which must exist between the differential invariants μ and u.

The first way is to notice that $\partial^2\mu + \mu\partial u - \bar{\partial} u$ is a *new* differential invariant [22, p. 212] (since it can be shown [21,22] that all (formal) derivatives of a differential invariant are again differential invariants) and it is the *only* linear combination involving derivatives of μ and the first order derivative of u *not* containing derivatives of Z of order ≥ 3. Hence it must be a rational function of $(\mu, \partial\mu, \bar{\partial}\mu, u)$ which happens to be here $-u\partial\mu$. Consequently we get the *compatibility condition* between μ and u

$$\partial^2\mu + \mu\partial u - \bar{\partial} u + u\partial\mu = 0 . \tag{3.11}$$

Secondly the problem can also be solved by considering the second order linear system of PDEs

$$(\bar{\partial} - \mu\partial)Z = 0, \qquad (\partial^2 - u\partial)Z = 0.$$

This system is *not* formally integrable and we have to follow the general procedure to transform it into a new formally integrable system in order to find the *integrability conditions.* Since it is a finite type system, this standard procedure [20] allows us to transform it into a first order system. Such a procedure yields *here* (a chance) to introduce a linear condition which will become a Frobenius type linear system. Elementary linear algebra leads to recovery of the differential condition (3.11).

The projective case. In a similar way, the projective complex analytic transformations can be identified through their differential invariants of order three [see example 2.9(ii)]. We are then interested in solving the third order non-linear system

$$\Phi \equiv \frac{\bar{\partial} Z}{\partial Z} = \mu , \qquad \Phi^{\mathrm{proj}} \equiv \{Z, z\} \equiv \frac{\partial^3 Z}{\partial Z} - \frac{3}{2}\left(\frac{\partial^2 Z}{\partial Z}\right)^2 = v . \tag{3.12}$$

Eliminating the jets of order four we look for the remaining terms. This leads to the following *compatibility condition*:

$$\partial^3\mu + 2v\partial\mu + \mu\partial v - \bar{\partial} v = 0. \tag{3.13}$$

However the corresponding system of PDEs now becomes non-linear in the second procedure

$$(\bar{\partial} - \mu\partial)Z = 0 , \qquad \partial Z \partial^3 Z - \tfrac{3}{2}(\partial^2 Z)^2 - v(\partial Z)^2 = 0.$$

Of course a direct study of formal integrability provides the same differential condition (3.13) as before, but now the system can only be transformed into a non-linear Frobenius type system. Indeed introducing the new jet variables $\sigma = Z_z$ for one-jets and $\tau = Z_{zz}$ for two-jets we get the following non-linear system:

$$\partial\sigma = \tau, \qquad \bar\partial\sigma = \mu\tau + \partial\mu\sigma,$$
$$\partial\tau = \tfrac{3}{2}\tau^2/\sigma + v\sigma, \qquad \bar\partial\tau = 2\partial\mu\tau + \partial^2\mu\sigma + \tfrac{3}{2}\mu\tau^2/\sigma + \mu v\sigma.$$

It is pure chance that the (rational) change of jet variables

$$(\sigma,\tau) \longmapsto (\psi,\varphi) : \sigma = \psi^{-2},\ \tau = -2\psi^{-3}\varphi$$

linearizes the above system and allows us to introduce a linear connection as in ref. [10]. So the corresponding SL$(2,\mathbb{C})$-flat bundle is that of the two-jet bundle over the Riemann surface X made from $(-\frac{1}{2},0)$-differentials.

Remark 3.2. Let φ be a holomorphic transformation, i.e. $\varphi \in \Gamma$. The transformation law under a holomorphic transformation at the *source* of the affine differential invariant Φ^{aff} provides the patching rules for constructing the natural bundle [cf. (2.17)]

$$\mathcal{F}^{\mathrm{aff}} : z' = \varphi(z) \Rightarrow u = u'\partial\varphi + \partial^2\varphi/\partial\varphi, \tag{3.14}$$

and a given section u of $\mathcal{F}^{\mathrm{aff}}$ will define an affine structure subordinate to the given complex structure on X.

Similarly, for the projective case, the change of the invariant differential Φ^{proj} induced by a holomorphic source transformation leads to the following natural bundle [cf. (2.19)]:

$$\mathcal{F}^{\mathrm{proj}} : z' = \varphi(z) \Rightarrow v = v'(\partial\varphi)^2 + \{\varphi, z\}; \tag{3.15}$$

a fixed section v of $\mathcal{F}^{\mathrm{proj}}$ will endow X with a projective complex structure.

Let us make a connection with the classical literature. The patching rules (3.14) and (3.15) are those considered by Gunning [28] for constructing such structures on a Riemann surface. But with our definition 2.10, we cannot call them affine or projective "connections" as Gunning did. As quoted in remark 2.11 they are *not* a splitting of a certain short exact sequence, but are actually geometrical objects, namely sections of certain natural bundles.

Remark 3.3. Since the affine group is a subgroup of the projective group one has the relation $v = \partial u - \frac{1}{2}u^2$, where u and v are defined in eqs. (3.10) and (3.12), respectively. The elimination of u between this last identity and eq. (3.11) provides the compatibility condition (3.13).

Remark 3.4. Dealing with the general *affine structure* (μ, u) or the general *projective structure* (μ, v) we may introduce on them the general scheme of the *formal Lie derivative.* The corresponding systems of infinitesimal Lie equations are now linear and of finite type. It follows that actually their integrability condition can be introduced in the game by means of a linear connection.

In an equivalent way the Spencer operator $D : \mathrm{R}_{q+1} \longrightarrow T^* \otimes \mathrm{R}_q$ now becomes an operator $D : \mathrm{R}_q \longrightarrow T^* \otimes \mathrm{R}_q$ because $\mathrm{R}_{q+1} = \mathrm{R}_q$ (the fact that the symbol $M_{q+1} = 0$ is equivalent to the finite type property). However, the integrability condition coming from the curvature may be of second order in the structure because of the appearence of certain structure constants [existence of some integrating factors as in example 2.14(iv)].

Example 3.5. Let us consider the pseudogroup defined by

$$\partial Z = 1,\ \bar{\partial} Z = 0 \Leftrightarrow Z = z + (a + \mathrm{i}b) \Leftrightarrow \begin{cases} y^1 = x^1 + a \\ y^2 = x^2 + b \end{cases} .$$

Then the corresponding equivalence problem becomes

$$\bar{\partial} Z / \partial Z = \mu, \qquad \partial Z = w,$$

and leads to the compatibility condition

$$\bar{\partial} w - \mu \partial w - w \partial \mu = 0,$$

as a way to saturate the first order jets. The corresponding structure is that of a Maurer–Cartan form or in other words $\mathcal{F} = T^* \times_X T^*$, and yields the well known Maurer–Cartan equations involving two structure constants.

4. The first non-linear Spencer sequence for the 2-d conformal pseudogroup

Our goal is to build the first non-linear Spencer sequence for the same pseudogroup Γ of complex analytic transformations that we have in mind in the particular dimensional case $n = 2$. It turns out as a very surprising result that the geometrically well defined formulas thus computed reproduce exactly those coming from the "gauging" procedure à la BRS given by ref. [11] for the Virasoro case.

Our strategy will be the following. Since the authors of ref. [11] use a bigraded differential algebra in which the curvature (in the sense of Maurer–Cartan, grading of forms) and the infinitesimal gauge variation (in the sense of BRS, ghost grading) are *mixed together* through the so called "russian formula" [1,2] the bigraded forms described in ref. [11] will be decomposed with respect to the ghost grading in order to exhibit the proper geometrical significance of the "gauge fields", which are viewed there as one-forms with values in some *infinite* Lie algebra, namely the maximal proper subalgebra w_2 of the Virasoro algebra. To save space, further details concerning ref. [11] are deferred to appendix B to which the reader is kindly referred.

Thus we assert that the components of ghost number *zero* of the vanishing bigraded curvature conditions worked out in ref. [11], formula (6) [#5] [or eq. (B.4)], are actually the compatiblity conditions $\tau_{q-1} = \overline{D}'\chi_q = 0$ (2.42) in the first non-linear Spencer sequence. Moreover the components with ghost numbers one and two will be related to the infinitesimal gauge variations (2.44) [in fact (2.47) as we shall see].

4.1. THE SPENCER SEQUENCE IN A HOLOMORPHIC REPRESENTATION

According to the results obtained in section 3, the pseudogroup Γ is defined by an involutive system $\mathcal{R}_1$ of finite Lie equations, see eqs. (3.3) or (3.8). Let us take the one-jet $f_1 \in \mathcal{R}_1$ to be infinitesimal,

$$\begin{aligned} f_1 : (z,\bar{z}) &\mapsto (z,\bar{z}; Z, \overline{Z}; Z_z, 0; 0, \overline{Z}_{\bar{z}}) \\ &= (z,\bar{z}; z + t\xi^z + \cdots, \bar{z} + t\xi^{\bar{z}} + \cdots, 1 + t\xi^z_z + \cdots, 0; 0, 1 + t\xi^{\bar{z}}_{\bar{z}} + \cdots), \end{aligned}$$

where $(\xi^z, \xi^{\bar{z}})$ denotes the components of a smooth vector field. Then the following linear systems of infinitesimal Lie equations R_{q+1} correspond to the various prolongations $\mathcal{R}_{q+1}$, $\forall q \geq 0$, see eq. (3.4):

$$\begin{aligned} &\mathrm{R}_{q+1} \subset J_{q+1}(T): \quad \xi^z_{\alpha+1_{\bar{z}}} = 0, \quad \xi^{\bar{z}}_{\alpha+1_z} = 0, \quad 0 \leq |\alpha| \leq q, \\ &\pi^{q+1}_0 \circ \xi_{q+1} = \xi_0 = (\xi^z, \xi^{\bar{z}}) \in T. \end{aligned} \tag{4.1}$$

This defines a linear system $\{\{q\}, \mathrm{R}_q\}$ over the positive integers q with the projections $\pi^{q+r}_q : \mathrm{R}_{q+r} \to \mathrm{R}_q$, $\forall r \geq 0$. Let us denote its projective limit by $\mathrm{R}_\infty = \mathrm{pr}\lim \mathrm{R}_q$. Let us recall the (restriction of the) first non-linear Spencer sequence [cf. (2.37)],

$$0 \longrightarrow \Gamma \xrightarrow{j_{q+1}} \mathcal{R}_{q+1} \xrightarrow{\overline{D}} T^* \otimes \mathrm{R}_q \xrightarrow{\overline{D}'} \wedge^2 T^* \otimes \mathrm{R}_{q-1}. \tag{4.2}$$

Following ref. [29] (first article, p. 433) for each q the vector bundle R_q can be complexified and it splits into $\mathrm{R}^{1,0}_q \oplus \mathrm{R}^{0,1}_q$ where $\mathrm{R}^{1,0}_q$ is a holomorphic vector bundle and $\mathrm{R}^{0,1}_q$ is the complex conjugate vector bundle. The projections π^{q+r}_q induce the projections $\mathrm{R}^{1,0}_{q+r} \to \mathrm{R}^{1,0}_q$, and in particular $\mathrm{R}^{1,0}_0 = T^{1,0}$ is the holomorphic tangent bundle of X and the projective limit $\mathrm{R}^{1,0}_\infty = \mathrm{pr}\lim \mathrm{R}^{1,0}_q$ is defined. So due to the complex decomposition a (real) section χ_q of $T^* \otimes \mathrm{R}_q$ splits into

$$\begin{aligned} &\chi_q = \chi^z_q \oplus \chi^{\bar{z}}_q, \quad \text{with } \chi^{\bar{z}}_q = \overline{\chi^z_q}, \\ &\chi^z_q \in \mathrm{R}^{1,0}_q : \chi^z_{\alpha+1_{\bar{z}},\bullet} = 0, \quad 0 \leq |\alpha| \leq q, \end{aligned} \tag{4.3}$$

#5 After correction of a misprint in the summation term, kindly communicated to us by R. Grimm.

thanks to (4.1) and $\chi_0^{\bar z} \in T^* \otimes T^{1,0}$. Next, in this holomorphic representation let us write the *intrinsic* formula's (2.42) for the various projective limits taking into account that $\chi_q^{\bar z} \in T^* \otimes \mathrm{R}_q^{1,0}$. Thanks to the decomposition (4.3), the general curvature conditions $\overline{D}'$ can be written down only for the components of $\chi_q^{\bar z}$ and we get successively the following *intrinsic* formulas with respect to the *complex analytic structure* on X:

$$\ldots q = 3 \begin{cases} q = 2 \begin{cases} q = 1 \, \{ \partial_z \chi^{z}_{,\bar z} - \partial_{\bar z} \chi^{z}_{,z} - \chi^{z}_{z,\bar z} - \chi^{z}_{,z} \chi^{z}_{z,\bar z} + \chi^{z}_{,\bar z} \chi^{z}_{z,z} = 0, \\ \partial_z \chi^{z}_{z,\bar z} - \partial_{\bar z} \chi^{z}_{z,z} - \chi^{z}_{zz,\bar z} - \chi^{z}_{,z} \chi^{z}_{zz,\bar z} + \chi^{z}_{,\bar z} \chi^{z}_{zz,z} = 0, \end{cases} \\ \partial_z \chi^{z}_{zz,\bar z} - \partial_{\bar z} \chi^{z}_{zz,z} - \chi^{z}_{zzz,\bar z} - \chi^{z}_{,z} \chi^{z}_{zzz,\bar z} \\ \qquad + \chi^{z}_{,\bar z} \chi^{z}_{zzz,z} - \chi^{z}_{z,z} \chi^{z}_{zz,\bar z} + \chi^{z}_{z,\bar z} \chi^{z}_{zz,z} = 0, \end{cases} \tag{4.4}$$

and so on, where for $q = 2$ the commutation relation (3.2) for the one-jets implies that the quadratic terms appearing in eq. (2.42b), $\chi^{r}_{l,i} \chi^{k}_{r,j} - \chi^{r}_{l,j} \chi^{k}_{r,i}$, vanish exactly as for the terms involving $\widetilde{M}^0$ in formula (9) of ref. [11].

Next, in a real representation substituting relation (2.40a), $\chi^{k}_{,i} = A^{k}_{i} - \delta^{k}_{i}$, into the general curvature formulae $\tau_{q-1} = \overline{D}' \chi_q = D\chi_q - \{\chi_q, \chi_q\}$, section of $\wedge^2 T^* \otimes \mathrm{R}_{q-1}$ [cf. (2.36)], we get for the various components of τ_{q-1} the following system of covariant PDEs, with $0 \leq |\alpha| \leq q-1$:

$$\tau^{k}_{\alpha,ij} = \underline{\underbrace{\partial_i \chi^{k}_{\alpha,j} - \partial_j \chi^{k}_{\alpha,i}}_{d} - A^{r}_{i} \chi^{k}_{\alpha+1_r,j} +} A^{r}_{j} \chi^{k}_{\alpha+1_r,i} + \cdots , \tag{4.5}$$

where dots represent terms containing only χ_p for $p \leq |\alpha|$ in a non-linear way. Notice that in each equation of this system the higher order term in the χ's which has been isolated is linear. The substitution (2.40a) *spoils* the separate intrinsic nature of $D\chi_q$ and of the quadratic terms $\{\chi_q, \chi_q\}$ in τ_{q-1} but the *whole* expression still remains covariant. Worse, it *breaks* the naturality of the Spencer operator and the algebraic bracket by reintroducing the usual exterior derivative d plus some quadratic terms [#6]. In particular, performing this splitting in complex coordinates, it is written as

$$\chi^{z}_{,z} = A^{z}_{z} - 1 , \quad \chi^{z}_{,\bar z} = A^{z}_{\bar z}, \tag{4.6}$$

and by substituting (4.6) in (4.4), taking into account that χ_q are *q-jet* valued one-forms, we obtain the following covariant formulas:

[#6] This is the natural way [22] to recover the usual Cartan notions of torsion and curvature whenever the second order jets vanish (affine pseudogroups).

$$\ldots\ \tau_2 \begin{cases} \tau_1 \begin{cases} \tau_0\ \{\partial_z A^{z}_{\bar z} - \partial_{\bar z} A^{z}_{z} - A^{z}_{z}\chi^{z}_{z,\bar z} + A^{z}_{\bar z}\chi^{z}_{z,z} = 0\,, \\ \partial_z \chi^{z}_{z,\bar z} - \partial_{\bar z}\chi^{z}_{z,z} - A^{z}_{z}\chi^{z}_{zz,\bar z} + A^{z}_{\bar z}\chi^{z}_{zz,z} = 0\,, \end{cases} \\ \partial_z \chi^{z}_{zz,\bar z} - \partial_{\bar z}\chi^{z}_{zz,z} - A^{z}_{z}\chi^{z}_{zzz,\bar z} + A^{z}_{\bar z}\chi^{z}_{zzz,z} \\ \qquad\qquad - \chi^{z}_{z,z}\chi^{z}_{zz,\bar z} + \chi^{z}_{z,\bar z}\chi^{z}_{zz,z} = 0\,. \end{cases} \tag{4.7}$$

Remark 4.1. Let us point out that the expression of τ_0 is recognized to be the torsion, but in the general curvature τ_1, since the quadratic terms in one-jets cancel out, there remain cross-terms which are products of *second* order jets with the matrix A and therefore cannot be identified as the usual Cartan curvature.

Finally, in this complex representation the first non-linear Spencer sequence (4.2) can be holomorphically decomposed as

$$0 \longrightarrow \Gamma \xrightarrow{j_{q+1}} \mathcal{R}_{q+1} \xrightarrow{\overline{D}^{1,0}} T^* \otimes \mathrm{R}^{1,0}_{q} \xrightarrow{\overline{D}'} \wedge^2 T^* \otimes \mathrm{R}^{1,0}_{q-1}\,, \tag{4.8}$$

where the operators are defined by

$$\begin{aligned} \overline{D}^{1,0} f_{q+1} &= (\overline{D} f_{q+1})^z = \chi^{z}_{q}\,, \\ \overline{D}'\chi^{z}_{q} &= \tau^{z}_{q-1} = D\chi^{z}_{q} - \{\chi^{z}_{q}, \chi^{z}_{q}\}\,, \end{aligned} \tag{4.9}$$

with of course $\overline{D}' \circ \overline{D}^{1,0} = 0$ and notice that the algebraic bracket is restricted to $\mathrm{R}^{1,0}_{q}$, see eq. (C.3) in appendix C.

Going further, we would like to recover the algebraic formulation as stated in ref. [11]. This will be achieved by some combinatorial change of jet coordinates and, as we shall see, it will be heavily related to the formal properties of the various prolongations R_{q+1}. For a fixed order q in jets let us first redefine the components of the section χ^{z}_{q} of $T^* \otimes \mathrm{R}^{1,0}_{q}$ by

$$\chi^{z}_{\alpha,\bullet} \equiv |\alpha|!\ \theta^{z}_{\alpha,\bullet}\,, \qquad \text{for } 0 \le |\alpha| \le q\,, \tag{4.10}$$

since the jet coordinate $\chi^{z}_{\alpha,\bullet}$ is related to the $|\alpha|$th derivative in some Taylor expansion. Therefore $\theta^{z}_{\alpha,\bullet}$ is rather considered as related to some power series and θ^{z}_{q} is another section of $T^* \otimes \mathrm{R}^{1,0}_{q}$. Having this in mind, for a fixed lenght $0 \le |\alpha| \le q-1$, the components of the general curvature (4.7) can be redefined as well:

$$\begin{aligned} F^{|\alpha|-1}_{zz} &\equiv \frac{1}{|\alpha|!}\,\tau^{z}_{\alpha,z\bar z} \\ &= \partial_z \theta^{z}_{\alpha,\bar z} - \partial_{\bar z}\theta^{z}_{\alpha,z} - (|\alpha| + 1)\theta^{z}_{\alpha+1_z,\bar z} - \frac{1}{|\alpha|!}\{\chi^{z}_{q}, \chi^{z}_{q}\}_{\alpha,z\bar z} \end{aligned}$$

$$= \partial_z \theta^z_{\alpha,\bar z} - \partial_{\bar z} \theta^z_{\alpha,z} - (|\alpha| + 1)\left(A^z_z \theta^z_{\alpha+1_z,\bar z} - A^z_{\bar z} \theta^z_{\alpha+1_z,z}\right) \qquad (4.11)$$

$$+ \sum_{|\lambda|=1}^{|\alpha|} \sum_{|\nu|=0}^{|\alpha|-1} \tfrac{1}{2}(|\lambda| - |\nu| - 1)\left(\theta^z_{\lambda,z}\theta^z_{\nu+1_z,\bar z} - \theta^z_{\lambda,\bar z}\theta^z_{\nu+1_z,z}\right)\delta^\alpha_{\lambda+\nu},$$

where we have used the combinatorial properties of the algebraic bracket restricted to R_q, see appendix C, eq. (C.6), and once more the change (4.6). Then let us set successively

$$\vartheta^{|\alpha|-1}_\bullet \equiv \theta^z_{\alpha,\bullet}, \quad \text{for } 1 \le |\alpha| \le q-1,$$
$$\vartheta^{-1}_\bullet \equiv A^z_\bullet, \quad \text{for } \alpha = 0. \qquad (4.12)$$

With $n = |\alpha| - 1$, $k = |\lambda| - 1$ and $l = |\nu|$ for $n \ge -1$ we obtain the two-form in terms of the exterior derivative,

$$F^n = \mathrm{d}\vartheta^n + \sum_{k,l=-1}^{n} \tfrac{1}{2}(k-l)\vartheta^k\vartheta^l\delta^n_{k+l}, \qquad (4.13)$$

which is exactly the two-form of ghost number zero of the bigraded curvature form $\widetilde{F}^n$ given in ref. [11], see also appendix B, eq. (B.4). Notice that the redefinition (4.12) *completely hides the jet nature* of the quantities at hand. Moreover the structure constants $(k-l)$, which are at first sight of an algebraic nature, come directly from the jet properties of $R^{1,0}_{q+1}$ as shown in appendix C. Moreover the explicit formulas for the operator $\overline{D}'$ in the Spencer sequence lead us to recover the key formulas of ref. [11]. The exterior system (4.13) labelled by the integer $n \ge -1$ is then equivalent to that expressed in terms of jets (4.7).

Assuming that we can solve the system (4.7), then the device of ref. [11] for writing compactly the successive curvature F^n can be reformulated in the following way: As in ref. [11] let us introduce a complex variable t and the one-form $\boldsymbol{\vartheta}$, which can be formally expanded in power series as

$$\boldsymbol{\vartheta} = \sum_{|\alpha|\ge 0} t^{|\alpha|}\,\vartheta^{|\alpha|-1} = A^z + \sum_{|\alpha|\ge 1} \frac{t^{|\alpha|}}{|\alpha|!}\chi^z_\alpha, \qquad (4.14)$$

by virtue of (4.10) and (4.12). So this simply shows that in fact we are concerned with formal Taylor expansion of the one-form $\boldsymbol{\vartheta}$ with values in $R^{1,0}_\infty$. For the curvature forms earlier defined in eq. (4.13) a two-form with values in $R^{1,0}_\infty$ can be similarly constructed,

$$\boldsymbol{F} = \sum_{|\alpha|\ge 0} t^{|\alpha|} F^{|\alpha|-1} = \sum_{|\alpha|\ge 0} \frac{t^{|\alpha|}}{|\alpha|!}\tau^z_\alpha. \qquad (4.15)$$

4.2. THE REAL RESOLUTION OF THE CURVATURE EQUATIONS

Let us now discuss the general resolution of the system (4.5) of PDEs. Suppose that A is a given arbitrary non-degenerate matrix, $\det A \neq 0$. Once A is fixed the system becomes linear. Starting with the equation $\tau_0 = 0$ (null torsion), we may then find a solution χ_1 of $\overline{D}'\chi_1 = \tau_0 = 0$. By induction we may find a section χ_q solution of $\tau_{q-1} = 0$. Assume that χ_{q-1} has been determined inductively, then at the next order q we have to solve (4.5) for $\chi^k_{\alpha+1}$, $|\alpha| = q-1$. However, it must be noticed that, contrary to $D\tau_{q-1}$ defined in (2.43), the direct computation of $\mathrm{d}\tau_{q-1}$ from (4.5) given by the system (with $0 \leq |\alpha| \leq q-1$)

$$\partial_k \tau^l_{\alpha,ij} + \overset{\hookrightarrow}{(i,j,k)} = A^r_k \underbrace{(\partial_i \chi^l_{\alpha+1_r,j} - \partial_j \chi^l_{\alpha+1_r,i})}_{\tau_q} + \overset{\hookrightarrow}{(i,j,k)} + \cdots ,$$

involves τ_q defined in eq. (4.5). That is to say, in a jet formulation taking the differential of τ_{q-1} is meaningless. The inductive step used by the authors of ref. [11] leading to an algebraic resolution of their curvature equations consits extactly in applying the exterior derivative to curvature forms and therefore *cannot be repeated here.* Overcoming this bad state of affairs consists in the elimination of the higher order jets implemented through a "twisting" by A. Let us set $\chi_q = \sigma_q \circ A$, that is in components $\chi^k_{\alpha,i} = A^r_i \sigma^k_{\alpha,r}$, $0 \leq |\alpha| \leq q$. Substituting this into eqs. (4.5) we easily get

$$\sigma^k_{\alpha+1_i,j} - \sigma^k_{\alpha+1_j,i} = B^r_i B^s_j (\partial_r \chi^k_{\alpha,s} - \partial_s \chi^k_{\alpha,r} + \cdots) , \quad 0 \leq |\alpha| \leq q-1 , \tag{4.16}$$

where $B = A^{-1}$ and dots stand for lower order jets. So solving the system in terms of σ_q is equivalent to solving it in χ_q since A is not degenerate. The left hand side of (4.16) is related to the algebraic Spencer δ-map [19–22]. Let us interpret this in our particular case. Recall that for any $\sigma_q \in T^* \otimes M_q$, where M_q is the symbol of R_q, we may define define the following map:

$$\delta : T^* \otimes M_{q+1} \longrightarrow \wedge^2 T^* \otimes M_q$$

by $(\delta\sigma_q)^k_\alpha = \mathrm{d}x^i \wedge \sigma^k_{\alpha+1_i}$, $0 \leq |\alpha| \leq q-1$, which in local coordinates is written as $(\delta\sigma_q)^k_{\alpha,ij} = \sigma^k_{\alpha+1_i,j} - \sigma^k_{\alpha+1_j,i}$. It is easy to check that δ is nilpotent, $\delta \circ \delta = 0$. So solving eq. (4.16) amounts to solving the Spencer δ-cohomology [21, p. 580], [22, p. 236]. We cannot insist on this technical step related to the symbol M_1 of R_1; we can just say that the study of the Spencer δ-cohomology for the symbol M_1 shows that the cohomology is trivial and that M_1 is involutive. (In fact in a general situation the solvability of the general curvature equations is equivalent to the two-acyclicity property of the symbol of the given algebroid with respect to the Spencer δ-cohomology, and involutive means n-acyclicity where $n = \dim X$.)

So thanks to this algebraic structure it is possible to solve the curvature equations (4.5) as a given system of PDEs without integrating. The price to pay in doing so is the loss of the compatibility condition property carried at the beginning by (4.5). Thus this leads to the problem of the local exactness of the first non-linear Spencer sequence at $T^* \otimes \mathrm{R}_q$, that is, given a section χ_q of $T^* \otimes \mathrm{R}_q$, does there exist a section f_{q+1} of $\mathcal{R}_{q+1}$ such that $\overline{D} f_{q+1} = \chi_q$? This is a problem which is soluble by mathematical analysis and is concerned with the elliptic property. Here, since the Cauchy–Riemann system is elliptic [29], the local exactness is insured.

Remark 4.2. The above procedure used in order to cancel out the higher order jets is just the way to pass from the first to the second Spencer sequence after twisting by A [21,22].

In order to show that this method supersedes the famous Cartan theorem for *ordinary* differential forms which underlies the algebraic induction used in ref. [11] let us do explicitly the case $q = 1$. Following step by step the above procedure, let χ_1 be a section of $T^* \otimes \mathrm{R}_1$ and consider its twist by A, $\chi_1 = \sigma_1 \circ A$. Then eq. (4.16) reads

$$\sigma^l_{r,s} - \sigma^l_{s,r} = B^i_r B^j_s (\partial_i A^l_j - \partial_j A^l_i) ,$$

which is just the coboundary condition with respect to the δ-map. Due to the dimension *two* this last condition is trivially satisfied (three-form) as can readily be seen from eq. (4.7). This insures that $\tau_0 = 0$ has a solution. Moreover computing directly $\mathrm{d}\tau_0$ and taking then $\tau_0 = 0$ we get [thanks to the commutation relations (3.2)]

$$A^r_k \tau^l_{r,ij} + \overset{\hookrightarrow}{(i,j,k)} = 0 ,$$

which is the crucial identity for the induction procedure of the construction shown in ref. [11] (cf. the technical lemma after formula (7)), but as said above this is meaningless in a jet theory. Recall also that we have made the assumption that A is an arbitrary non-degenerate matrix. The arbitrariness must therefore be analyzed since there are no differential equations constraining the matrix A. This turns out to be one of the most important points of this part.

4.3. THE LINK WITH THE COMPLEX STRUCTURE μ

Heretofore in dealing with the Spencer sequence we have *not* spoken about the complex structure μ which is the main ingredient in the construction given by ref. [11]. How should one introduce μ in the Spencer sequence while it lies in the Janet sequence? The answer will be furnished by the analysis of the previous section 3 as applied in the present case thanks to the main final theorem 2.20 of section 2.

Recall that any f_{q+1} belonging to the qth prolongation $\mathcal{R}_{q+1}$ [cf. (3.4)] of $\mathcal{R}_1$ has projection

$$\pi_1^{q+1} \circ f_{q+1} = f_1 \in \mathcal{R}_1 , \quad f_1 = \begin{pmatrix} Z_z & 0 \\ 0 & \overline{Z}_{\bar{z}} \end{pmatrix} , \quad Z_z \neq 0 ,$$

and that $\pi_0^1 \circ f_1 = f \in \mathrm{aut}(X)$ is an *arbitrary* diffeomorphism preserving the orientation. Thanks to the definition (2.41) of the matrix A we have

$$A_z^z(z,\bar{z}) = \frac{1}{Z_z} \partial_z Z , \quad A_{\bar{z}}^z(z,\bar{z}) = \frac{1}{Z_z} \partial_{\bar{z}} Z \tag{4.17}$$

(of course $A_z^z \neq 0$ by construction) and from expression (3.6) of the differential invariant of order one for Γ we get the following equality for $f \in \mathrm{aut}(X)$:

$$(z,\bar{z}) \overset{f}{\longmapsto} (Z,\overline{Z}) , \quad \Phi(j_1(f)(z,\bar{z})) \equiv \frac{\partial_{\bar{z}} Z}{\partial_z Z} = \frac{A_{\bar{z}}^z(z,\bar{z})}{A_z^z(z,\bar{z})} , \tag{4.18}$$

or directly in terms of the tensor J according to remark 3.1,

$$\Phi_j^i \equiv (J_0)_l^k(y) \frac{\partial y^l}{\partial x^j} \frac{\partial x^i}{\partial y^k} = (A^{-1})_r^i(x)(J_0)_s^r(x) A_j^s(x) , \tag{4.19}$$

which are two expressions for the same differential invariant of order *one* according to the complex or real formulation. This is an application of the last remark 2.22 of section 2. Also formula (4.18) [or (4.19)] *connects* together the non-linear Janet sequence and the non-linear Spencer sequence and it is the *only way* to relate them because the differential invariant is of order *one*.

Now solving the equivalence problem (3.9) means that for a *given complex structure* μ, $|\mu| < 1$, *on the source* seeking for all sections $f_1 \in \mathcal{R}_1$ such that $\pi_0^1 \circ f_1 = f \in \mathrm{aut}(X)$ is a solution of the equivalence problem. We obtain the important (pointwise) relation

$$\mu_{\bar{z}}^z(z,\bar{z}) = j_1(f)^{-1}(0)(z,\bar{z}) = \frac{\partial_{\bar{z}} Z}{\partial_z Z} = \frac{A_{\bar{z}}^z(z,\bar{z})}{A_z^z(z,\bar{z})} \tag{4.20a}$$

$$\Longleftrightarrow \qquad J_\mu(z,\bar{z}) = A^{-1}(z,\bar{z}) J_0(z,\bar{z}) A(z,\bar{z}) . \tag{4.20b}$$

Due to the dimension two there is *no* compatibility for μ and thus A is *free* of differential conditions. Apart from the condition $|A_{\bar{z}}^z / A_z^z| < 1$, A is *arbitrary* since μ is. Therefore the procedure for the inductive resolution of (4.5) described above can be applied.

Remark 4.3. This illustrates the delicate result explained in the proof of theorem 2.20. It is clear that the Spencer sequence is relative to the complex structure $\mu = 0$. But through the equivalence problem the complex structure $\mu = 0$ is put on the *target* while the complex structure μ lies on the source. Somehow we might say that the Spencer sequence is built for the target and the gauge theory sits on the source.

Accordingly, a finite (infinitesimal) gauge transformation of χ_0 (A) induces a natural transformation [see eq. (3.7)] (Lie derivative) of μ. Let us be more explicit about the infinitesimal gauge transformation. Recall that an infinitesimal gauge transformation, see definition 2.18, is induced by a *section* ξ_1 of the algebroid R_1, cf. (4.1). Taking the variation (2.49a) of χ_0 and inserting it in the identity (2.40a) we can then write down the infinitesimal gauge transformation of the matrix A in complex coordinates with the notation $(\xi\cdot\partial) \equiv \xi^z\partial_z + \xi^{\bar z}\partial_{\bar z}$ for vector fields,

$$\begin{aligned}\delta A^z_z &= (\xi\cdot\partial)A^z_z + A^z_z\partial_z\xi^z + A^{\bar z}_z\partial_z\xi^{\bar z} - \xi^z_z A^z_z\,,\\ \delta A^{\bar z}_{\bar z} &= (\xi\cdot\partial)A^{\bar z}_{\bar z} + A^z_{\bar z}\partial_{\bar z}\xi^z + A^{\bar z}_{\bar z}\partial_{\bar z}\xi^{\bar z} - \xi^z_z A^{\bar z}_{\bar z}\,,\end{aligned}$$

which depends on one-jets. Next with (4.20) a direct computation gives

$$\begin{aligned}\mathcal{L}(\xi)\mu^z_{\bar z} &= \delta\,(A^z_{\bar z}/A^z_z)\\ &= (\xi\cdot\partial)\mu^z_{\bar z} + \partial_{\bar z}\xi^z + \mu^{\bar z}_{\bar z}\partial_{\bar z}\xi^{\bar z} - \mu^z_{\bar z}(\partial_z\xi^z + \mu^{\bar z}_{\bar z}\partial_z\xi^{\bar z})\,, \quad (4.21)\end{aligned}$$

where in the course of the computation *all* the jet dependence coming from δA have cancelled out. It is rather surprising that the Lie derivative of μ (usually induced by a diffeomorphism) is directly computed from the infinitesimal gauge transformation of some objects depending only on $\mathcal{R}_1$.

It is also of interest to perform the gauge transformation (2.49b) for χ_0 induced by the isomorphic change of "generators" (2.46) in R_{q+1}. For convenience in order to relate the notation with some previous works [3,7,8], let us rewite (2.46) at order zero as

$$\Xi \equiv \xi' = \xi + \chi_0(\xi) = A(\xi)\,, \qquad \det A \neq 0\,. \qquad (4.22)$$

It is easily checked that $\Xi_1 \in R_1$. Then we get the well defined expression

$$\mathcal{L}(\xi)\mu^z_{\bar z} = \frac{1}{A^z_z}\left(\partial_{\bar z}\Xi^z - \mu^z_{\bar z}\partial_z\Xi^z + \frac{\Xi^z}{A^z_z}(\partial_z A^z_{\bar z} - \partial_{\bar z}A^z_z)\right)\,, \qquad (4.23)$$

where the dependence on A is quite normal due to the change (4.22). With respect to this change of generators the dependence of the variation on the lift χ_1 of χ_0, cf. (2.47), implies the use of the compatibility condition τ_0 in (4.7) and thus derivatives of A occur in doing so.

Let us say a little bit more about this matrix A. The reader could identify it with the so called "zweibein" as introduced in refs. [7–9] thanks to relation (4.20). The ambiguity in the determination of A can be lifted in the following way. Let us suppose that an ${}^0\!A$ fulfilling the pointwise relation (4.20) has been found and let $A = \beta\,{}^0\!A$ be another solution of (4.20), where the matrix β has to be determined. By assumption on A and thanks to (4.20b) we must have $[J_0,\beta](z,\bar z) = 0$. The general form for such a matrix commuting with J_0 is (see, e.g., ref. [26])

$$\beta(z,\bar{z}) = \begin{pmatrix} b(z,\bar{z}) & 0 \\ 0 & \bar{b}(z,\bar{z}) \end{pmatrix},$$

where $b(z,\bar{z})$ is a non-vanishing complex valued function. Thus any two by two matrix valued function $A = \beta\, {}^0\!A$ on X is the general solution of the equivalence problem (4.20) whenever the matrix ${}^0\!A$ has been given. To get such a special solution ${}^0\!A$ of (4.20) we can read from (3.5) (recall that $|\mu_{\bar{z}}^{z}| < 1$)

$${}^0\!A = \begin{pmatrix} 1 & \mu \\ \bar{\mu} & 1 \end{pmatrix}, \tag{4.24}$$

and hence the general form of A is written as

$$A(z,\bar{z}) = \begin{pmatrix} b & b\mu \\ \bar{b}\bar{\mu} & \bar{b} \end{pmatrix}(z,\bar{z}). \tag{4.25}$$

4.4. THE "GAUGE FIXING" AND THE SPENCER OPERATOR

Let us now use in more detail the holomorphic representation already exhibited above. Recall that the first non-linear Spencer sequence (4.2) starts with

$$0 \longrightarrow \Gamma \xrightarrow{j_{q+1}} \mathcal{R}_{q+1} \xrightarrow{\bar{D}} T^* \otimes \mathrm{R}_q \tag{4.26}$$

with $\bar{D}f_{q+1} = f_{q+1}^{-1} \circ j_1(f_q) - \mathrm{id}_{q+1}$ and a transformation f belongs to the pseudogroup Γ if and only if $f_{q+1} = j_{q+1}(f)$ since $\bar{D} \circ j_{q+1} = 0$. According to the complex formulation of the first non-linear Spencer sequence (4.8), we are in fact concerned only with the holomorphic vector bundle $\mathrm{R}_q^{1,0}$. We can rather write (4.26) as

$$0 \longrightarrow \Gamma \xrightarrow{j_{q+1}} \mathcal{R}_{q+1} \xrightarrow{\bar{D}^{1,0}} T^* \otimes \mathrm{R}_q^{1,0} \tag{4.27}$$

with, of course, $\bar{D}^{1,0} \circ j_{q+1} = 0$. The differential operator $\bar{D}^{1,0}$ splits into the sum of two operators, namely

$$\bar{D}^{1,0} = \bar{D}' + \bar{D}'', \quad \begin{cases} \bar{D}' : \mathcal{R}_{q+1} \longrightarrow T^{*1,0} \otimes \mathrm{R}_q^{1,0}, \\ \bar{D}'' : \mathcal{R}_{q+1} \longrightarrow T^{*0,1} \otimes \mathrm{R}_q^{1,0}, \end{cases} \tag{4.28}$$

where $\bar{D}'$ and $\bar{D}''$ are of types (1,0) and (0,1), respectively, according to the usual splitting $\mathrm{d} = \partial + \bar{\partial}$ of the exterior differential operator d. Taking the image in $\mathrm{R}_q^{1,0}$ we have

$$\chi_q^z = \bar{D}^{1,0} f_{q+1} = \bar{D}' f_{q+1} + \bar{D}'' f_{q+1} \tag{4.29}$$

with, for $0 \leq |\alpha| \leq q$,

$$(\overline{\mathcal{D}}' f_{q+1})_{\alpha,z} = \chi^z_{\alpha,z}, \qquad (\overline{\mathcal{D}}' f_{q+1})_{\alpha,\bar{z}} = 0, \tag{4.30a}$$

$$(\overline{\mathcal{D}}'' f_{q+1})_{\alpha,z} = 0, \qquad (\overline{\mathcal{D}}'' f_{q+1})_{\alpha,\bar{z}} = \chi^z_{\alpha,\bar{z}}. \tag{4.30b}$$

We can thus use the fundamental inductive formula [22, p. 229] defining in local coordinates the section χ^z_q of $T^* \otimes \mathrm{R}^{1,0}_q$ in order to compute $\overline{\mathcal{D}}' f_{q+1}$ and $\overline{\mathcal{D}}'' f_{q+1}$, respectively, that is [with eq. (3.4)],

$$\overline{\mathcal{D}}' f_{q+1} : \sum_{|\lambda|=0}^{|\alpha|} \sum_{|\nu|=0}^{|\alpha|} \frac{(|\lambda| + |\nu|)!}{|\lambda|!\,|\nu|!} Z_{\lambda+1_z} \chi^z_{\nu,z}\, \delta^\alpha_{\lambda+\nu} = \partial_z Z_\alpha - Z_{\alpha+1_z}, \tag{4.31a}$$

$$\overline{\mathcal{D}}'' f_{q+1} : \sum_{|\lambda|=0}^{|\alpha|} \sum_{|\nu|=0}^{|\alpha|} \frac{(|\lambda| + |\nu|)!}{|\lambda|!\,|\nu|!} Z_{\lambda+1_z} \chi^z_{\nu,\bar{z}}\, \delta^\alpha_{\lambda+\nu} = \partial_{\bar{z}} Z_\alpha, \tag{4.31b}$$

where in the right hand sides we recover the components of the (non-linear) Spencer operator D, see (2.31).

Let us now concentrate on the kernel of the operator $\overline{\mathcal{D}}'$. The $\overline{\mathcal{D}}'$ operator is only concerned with the components $f^Z_\alpha(z,\bar{z}) = Z_\alpha$, $0 \leq |\alpha| \leq q+1$, of a section f_{q+1} of $\mathcal{R}_{q+1}$. From (4.31a) it is readily found that for each positive integer q

$$\operatorname{Ker} \overline{\mathcal{D}}' = \{ f_{q+1} \in \mathcal{R}_{q+1} \,/\, Z_\alpha = \partial_z^{|\alpha|} Z \ ,\ 0 \leq |\alpha| \leq q+1 \}, \tag{4.32}$$

but this does *not* mean at all that $\pi^{q+1}_0 \circ f_{q+1} = f$ belongs to Γ since $f \in \operatorname{aut}(X)$ has no constraint on $\partial_{\bar{z}} Z$ because $\overline{\mathcal{D}}'$ is only a part of the operator $\overline{D}$.

In particular for $q = 1$, a section $f_1 \in \mathcal{R}_1$ is in the kernel of $\overline{\mathcal{D}}'$ if and only if $\partial_z Z = Z_z$, or in other words, see eqs. (4.6), (4.17) and (4.25),

$$\overline{\mathcal{D}}' f_1 = \chi^z_{,z} = 0 \iff A^z_z = b \equiv 1 . \tag{4.33}$$

Demanding that the section $f_1 \in \operatorname{Ker} \overline{\mathcal{D}}'$ imposes the condition on the non-vanishing complex function b that it be the constant function 1, eq. (4.23) takes the form

$$\mathcal{L}(\xi)\mu^z_{\bar{z}} = \partial_{\bar{z}} \Xi^z - \mu^z_{\bar{z}} \partial_z \Xi^z + \Xi^z \partial_z \mu^z_{\bar{z}} , \qquad \text{with } \Xi^z = \xi^z + \mu^z_{\bar{z}} \xi^{\bar{z}} , \tag{4.34}$$

which is exactly the BRS transformation for μ used in ref. [11], or see (B.1). The change of generators $\Xi^z = \xi^z + \mu^z_{\bar{z}}\xi^{\bar{z}}$ first found in ref. [30] reflects the complex structure of $\operatorname{aut}(X)$ relative to the complex structure μ and is deeply related to the so called chiral splitting (or holomorphic factorization) property of 2-d conformal models [3–7,30].

The condition $b = 1$ which occurs here for a purely mathematical reason provides in our opinion the correct explanation for the suitable "gauge fixing" performed in ref. [11].

The component of ghost number zero of the *algebraic connection* [2] $\widetilde{M}^{-1}$ can readily be identified with our "zweibein" matrix A given by (4.25) in order to have a well defined jet formulation and apply the recursive construction. This explains the *a priori* choice of $\widetilde{M}^{-1}$ made in ref. [11].

But we said that the Spencer sequence for the Cauchy–Riemann system is locally exact, that is, we are *sure* that all solutions χ^z_q of the curvature equations (4.7) are of the form $\chi^z_q = \overline{D}^{1,0} f_{q+1}$ with $f_{q+1} \in \mathcal{R}_{q+1}$. According to this we may say that the *"fields"* χ^z_q come from *"potentials"* f_{q+1}. [#7] Using eqs. (4.31) for each $q \geq 0$ the components of the section χ^z_q are thus found inductively.

Starting with a section $f_1 \in \operatorname{Ker} \overline{\mathcal{D}}'$, i.e. $\partial_z Z = Z_z$, which corresponds to a "gauge fixing" in the "space of potentials", see eq. (4.24), eqs. (4.6) read

$$\chi^z_{,z} = {}^0A^z_z - 1 = 0\,, \qquad \chi^z_{,\bar z} = {}^0A^z_{\bar z} = \mu^z_{\bar z}\,,$$

and they imply at order $q = 1$

$$\chi^z_{z,z} = (\partial_z Z_z - {}^0A^z_z Z_{zz})/Z_z = (\partial^2_z Z - Z_{zz})/\partial_z Z\,, \tag{4.35a}$$

$$\begin{aligned}\chi^z_{z,\bar z} = (\partial_{\bar z} Z_z - {}^0A^z_{\bar z} Z_{zz})/Z_z &= (\partial_z \partial_{\bar z} Z - \mu^z_{\bar z} Z_{zz})/\partial_z Z \\ &= \partial_z \mu^z_{\bar z} + \mu^z_{\bar z} \chi^z_{z,z}\,,\end{aligned} \tag{4.35b}$$

where relation (4.20a) has been used to obtain the last equality in (4.35b). Moreover $\chi^z_{z,z}$ appears as a free parameter. These components are the general solutions of the equation τ_0 in (4.7),

$$\partial_z \mu^z_{\bar z} - \chi^z_{z,\bar z} + \mu^z_{\bar z} \chi^z_{z,z} = 0\,, \tag{4.36}$$

which is exactly the ghost number zero part (B.6a) in appendix B but with the jet geometrical meaning in addition. Notice that a particular solution for this equation is

$${}^0\chi^z_{z,z} = 0\,, \qquad {}^0\chi^z_{z,\bar z} = \partial_z \mu^z_{\bar z}\,, \tag{4.37}$$

but the measurement of the distance between the general section χ^z_1 and this special section ${}^0\chi^z_1 : \ (z, \bar z) \mapsto (z, \bar z, \mu^z_{\bar z}, \partial_z \mu^z_{\bar z})$ is achieved by the operator $\overline{\mathcal{D}}'$ by demanding that the section f_2 belongs to its kernel. This ambiguity computed here without the exterior calculus method is exactly that resulting from the computation made in appendix B, eq. (B.7). Indeed we have from eqs. (4.35)

$$\chi^z_1 = \partial_z \mu^z_{\bar z} \mathrm{d}\bar z + \chi^z_{z,z}(\mathrm{d}z + \mu^z_{\bar z} \mathrm{d}\bar z) = {}^0\chi^z_1 + \chi^z_{z,z}\, {}^0\!\mathcal{A}^z\,.$$

Performing a gauge transformation (2.49d) with $\Xi^z_2 \in \mathrm{R}^{1,0}_2$ we obtain

$$\begin{aligned}\delta\chi^z_{z,z} &= \partial_z \Xi^z_z + \chi^z_{zz,z} \Xi^z - A^z_z \Xi^z_{zz}\,, \\ \delta\chi^z_{z,\bar z} &= \partial_{\bar z} \Xi^z_z + \chi^z_{zz,\bar z} \Xi^z - A^z_{\bar z} \Xi^z_{zz}\,,\end{aligned}$$

[#7] This interpretation largely differs from that adopted in classical gauge theory, see eq. (2.48); for mechanical motivations and mathematical details, see ref. [22].

and thus by using eq. (4.7) for τ_1 the jets of order two occurring in the above variations are eliminated and we get from (4.36)

$$\begin{aligned}\delta(\partial_z\mu_{\bar z}^z) &= \partial_{\bar z}\Xi_z^z + \Xi^z(\partial_z\chi_{z,\bar z}^z - \partial_{\bar z}\chi_{z,z}^z)\\ &\quad -\mu_{\bar z}^z\partial_z\Xi_z^z - \chi_{z,z}^z(\partial_{\bar z}\Xi^z - \mu_{\bar z}^z\partial_z\Xi^z + \Xi^z\partial_z\mu_{\bar z}^z)\,. \qquad (4.38)\end{aligned}$$

Also the difference for this variation to be exactly $\delta\,{}^0\chi_1^z = \partial\mathcal{L}(\Xi)\mu$ is computed through the Spencer operator $\mathcal{D}'$ of type (1,0), see ref. [29], which is the linearized version of the operator $\overline{\mathcal{D}}'$. That is, if f_2 is in the kernel of $\overline{\mathcal{D}}'$ then Ξ_1^z will belong to Ker $\mathcal{D}'$. Indeed, by construction of Ξ_1^z through the isomorphism (2.46) we have $\mathcal{D}'\Xi_1^z = \mathcal{D}'\xi_1^z + \mathcal{D}'\chi_1^z(\xi)$. In components

$$\partial_z\Xi^z - \Xi_z^z = \partial_z\xi^z - \xi_z^z + \partial_z(\mu_{\bar z}^z\xi^{\bar z}) - \partial_z\mu_{\bar z}^z\xi^{\bar z} = 0\,,$$

where the required value of ξ_z^z is read from eq. (4.21). So the choice of such a f_2 implies $\Xi_z^z = \partial_z\Xi^z$ and $\chi_{z,z}^z = 0$. Conversely, since ξ_z^z is known from eq. (4.21), imposing $\mathcal{D}'\Xi_1^z = 0$ implies $f_2 \in \operatorname{Ker}\overline{\mathcal{D}}'$. In both cases, the gauge variation (4.38) becomes exactly $\partial\mathcal{L}(\Xi)\mu$.

To be more general, let us proceed by induction on q. Looking back at (4.31a) and since the section $f_{q+1} \in \mathcal{R}_{q+1}$ is invertible ($Z_z \neq 0$) suppose that for $0 \le |\alpha| \le q$,

$$\begin{aligned}\chi_{\alpha,z}^z \equiv 0 \;&\Leftrightarrow\; Z_{\alpha+1_z} = \partial_z Z_\alpha \Leftrightarrow\; Z_{\alpha+1_z} = \partial_z^{|\alpha|+1}Z\\ &\Leftrightarrow\; f_{q+1} \in \operatorname{Ker}\overline{\mathcal{D}}' \;\Rightarrow\; \chi_{\alpha,\bar z}^z \equiv \partial_z^{|\alpha|}\mu_{\bar z}^z, \qquad (4.39)\end{aligned}$$

by definition, see (4.32) and the resolution of (4.31b). In order words, we have up to order q

$$\chi_q^z = {}^0\chi_q^z = \overline{D}^{1,0}f_{q+1} = \overline{\mathcal{D}}''f_{q+1}\,, \quad \text{with } f_{q+1} \in \operatorname{Ker}\overline{\mathcal{D}}'\,. \qquad (4.40)$$

At order $q+1$ for $|\alpha| = q+1$ we have by the assumption (4.40) and from (4.31)

$$\begin{aligned}\chi_{\alpha,z}^z &= \frac{1}{\partial_z Z}(\partial_z^{|\alpha|+1}Z - Z_{\alpha+1_z})\,,\\ \chi_{\alpha,\bar z}^z &= \frac{1}{\partial_z Z}\Big(\partial_{\bar z}\partial_z^{|\alpha|}Z - \mu_{\bar z}^z Z_{\alpha+1_z} \qquad (4.41)\\ &\quad - \sum_{|\lambda|=0}^{|\alpha|}\sum_{|\nu|=0}^{|\alpha|}\frac{(|\lambda|+|\nu|)!}{|\lambda|!\,|\nu|!}\partial_z^{|\lambda|+1}Z\,\partial_z^{|\nu|}\mu_{\bar z}^z\,\delta_{\lambda+\nu}^\alpha\Big)\\ &= \partial_z^{|\alpha|}\mu_{\bar z}^z + \mu_{\bar z}^z\chi_{\alpha,z}^z\,,\end{aligned}$$

which yields

$$\chi_{\alpha,z}^z\,\mathrm{d}z + \chi_{\alpha,\bar z}^z\,\mathrm{d}\bar z = \partial_z^{|\alpha|}\mu_{\bar z}^z\,\mathrm{d}\bar z + \chi_{\alpha,z}^z\,{}^0\!A^z\,. \qquad (4.42)$$

Then *imposing* that the section $f_{q+2} \in \mathcal{R}_{q+2}$ such that

$$\pi_{q+1}^{q+2}\circ f_{q+2} = f_{q+1} \in \operatorname{Ker}\overline{\mathcal{D}}'\,, \qquad \overline{D}^{1,0}f_{q+2} = \chi_{q+1}^z\,,$$

must belong to Ker $\overline{\mathcal{D}}'$ we find

$$\chi^z_{q+1} = {}^0\chi^z_{q+1} = \partial_{q+1}(\mu) .$$

In conclusion, for any section (potential) f_{q+1} which belongs to Ker $\overline{\mathcal{D}}'$ ("gauge fixing" at each order) the image (field) $\chi^z_q = \overline{\mathcal{D}}^{1,0} f_{q+1}$ defined in (4.29) is the particularly simple section ${}^0\chi^z_q = \partial_q(\mu)$ (where ∂_q means that only derivatives with respect to z are performed in the differential operation j_q) is a q-jet of a section of the natural bundle $\mathcal{F}$ and belongs to $J_q(\mathcal{F})$. It can be shown [21,22] that, since μ is geometrical object for $\mathcal{R}_1$, $j_q(\mu)$ will be a geometrical object for $\mathcal{R}_{q+1}$ (and here thanks to the complex formulation $\partial_q(\mu)$ will be a geometrical object for $\mathcal{R}^{1,0}_{q+1}$). The gauge variation of such a section is then

$$\delta\, {}^0\chi^z_q = \delta\partial_q(\mu) = \partial^q \mathcal{L}(\Xi)\mu . \tag{4.43}$$

Choosing these special sections, and passing to the projective limit the general $\mathrm{R}^{1,0}_\infty$ valued one-form (4.14) becomes

$$\begin{aligned} {}^0\boldsymbol{\vartheta} &= \mathrm{d}z + \mathrm{d}\bar{z}\left(\mu^z_{\bar{z}} + \sum_{|\alpha|\geq 1} \frac{t^{|\alpha|}}{|\alpha|!}\, \partial_z^{|\alpha|}\mu^z_{\bar{z}}\right) \\ &= \exp\{t\,\partial_z\}\,(\mathrm{d}z + \mu^z_{\bar{z}}\,\mathrm{d}\bar{z}) = \exp\{t\,\partial_z\}\,{}^0A^z , \end{aligned}$$

which is exactly the zero ghost number part of the algebraic equation (13) worked out in ref. [11].

To recapitulate how in fact the Spencer theory and ref. [11] are intermingled, let us make some comments. Jet theory seems to be appropriate for treating in a more satisfactory geometrical way an *infinite* dimensional symmetry group. Then, once the geometric structure of a jet theory has been identified with respect to a general framework for dealing with Lie pseudogroups, the role of "gauge fields" is played in the useful holomorphic representation by the sections χ^z_q which are one-forms with values in some Lie algebroids $\mathrm{R}^{1,0}_q$. Moreover by virtue of the local exactness of the first non-linear Spencer sequence, these "gauge fields" come from "potentials" and a special choice of those potentials ("gauge fixing") gives rise to special fields. The space $\Theta \subset T^{1,0}$ of solutions of $\mathrm{R}^{1,0}_1$ is a sheaf of infinite Lie algebras. Intuitively the various generators of the maximal Lie subalgebra denoted by w_2 in ref. [11] can be chosen according to the comments at the end of appendix C. However this point remains to be clarified especially if a representation of w_2 is desired.

Let us end by saying that the BRS operation gives rise to an *algebraic* process for computing faster the successive infinitesimal gauge variations (4.43) at each jet order and the ambiguity arising at each order, see eq. (4.42) and also appendix B, is related to the question whether we are concerned with either a section of jet bundle or jet of section, and this is controlled by the Spencer operator $\mathcal{D}'$.

5. Conclusion

The approach described in the present paper and its application to 2-d conformal geometry call for the following short comment. Our motivation in writing up this paper was to provide some explanations in order to clarify the geometrical situation found in this topic of mathematical physics without completely answering the many questions arising from refs. [10] and [11]. Many aspects remain to be addressed and more detailed analysis is left to future investigations. However, we emphasize that with the help of concepts coming directly from jet theory and which cannot be otherwise introduced, we were able to link these two geometrical formulations of 2-d conformal field theories and see where they stand with respect to a general mathematical framework. As an explicit application, the Cauchy–Riemann system of PDEs sets a good example where terms involving *jets of order two* play a role of prominent importance (see remark 4.1).

The analysis presented in the third part works at level $n = 2$ of $\mathrm{SL}(n, \mathbb{C})$ and explains the construction given by ref. [10] at this level. But the investigation for using this formal theory to the study of the construction of W-algebras with their underlying geometry has not yet been achieved.

On the contrary, the computation in section 4 of the non-linear Spencer sequence gives rise to a precise and natural understanding of the algebraic generation in the Virasoro case built in ref. [11]. Let us point out that the main advantage of the present approach is to restrict a general formal theory to a specific example rather than developing a (good) "intuitive" construction concerned with exterior calculus and built up for a very particular case. Moreover due to the strong similarity of the construction worked out in ref. [11] for "gauging" $w_{1+\infty}$ algebras it suggests that the Spencer sequence ought to be also constructed for that case. But this requires finding first a (linear) system of PDEs (if one exists!) generating these $w_{1+\infty}$ algebras.

Jet theory supplies an appropriate setting for obtaining new information of a geometric nature. The point to emphasize is that *considering a given* Lie pseudogroup [#8] either the Janet or the Spencer sequences can be constructed giving rise to different geometric concepts and results. It conveys the idea that this formulation might offer a unifying geometrical framework for going much further in the investigation of gauge theories, especially when the symmetry group is infinite, and that the geometry emerging from non-linear Spencer sequence(s) might cast some new lights on the geometric structure of such theories.

The first author (S.L.) is deeply indebted to the second author (J.-F.P.) for introducing him little by little to the Lie pseudogroup technique and he also

[#8] The spirit of this approach is in complete agreement with the starting point of view of ref. [11], which was already considered by E. and F. Cosserat in 1909 for Continuum Mechanics [22].

wishes to thank R. Grimm for instructive discussions about the work [11] and for private communications. We are grateful to the referee for suggestions and constant encouragements in the completion of this work.

Appendix A

In this appendix the proof of theorem 2.19 is displayed. So performing two infinitesimal gauge transformations (2.44), the former with respect to a section ξ_{q+1} and the latter with respect to η_{q+1}, and then antisymmetrizing, the variation of χ_q reads

$$L(j_1(\eta_{q+1}))\big(L(j_1(\xi_{q+1}))\chi_q + D\xi_{q+1}\big) - (\eta \leftrightarrow \xi)\,.$$

Now evaluating this variation on a vector field $\zeta \in T$, and taking care of the splitting of the action of the formal Lie derivative as stated in definition 2.18, eq. (2.45), we obtain

$$\begin{aligned}&L(\eta_{q+1})\big(L(\xi_{q+1})i(\zeta) - i([\xi,\zeta])\big)\chi_q\\ &\quad -\big(L(\xi_{q+1})i([\eta,\zeta]) - i([\xi,[\eta,\zeta]])\big)\chi_q\\ &\quad +\big(L(\eta_{q+1})i(\zeta) - i([\eta,\zeta])\big)D\xi_{q+1} - (\eta \leftrightarrow \xi)\,.\end{aligned}$$

Using the algebraic relation [20, p. 384], [22, p. 206]

$$[L(\eta_{q+1}), L(\xi_{q+1})] = L([\eta_{q+1},\xi_{q+1}]),$$

and the definition of the action of $L(j_1(\xi_{q+1}))$ on $T^* \otimes \mathrm{R}_q$, the evaluation becomes

$$\begin{aligned}i(\zeta)L(j_1([\eta_{q+1},\xi_{q+1}]))\chi_q &+ \big(L(\eta_{q+1})i(\zeta) - i([\eta,\zeta])\big)D\xi_{q+1}\\ &-\big(L(\xi_{q+1})i(\zeta) - i([\xi,\zeta])\big)D\eta_{q+1}\,.\end{aligned}$$

Next since the action of the formal Lie derivative $L(\xi_{q+1})$ on $J_q(T) \supset \mathrm{R}_q$ is defined by [20, p. 383]

$$\begin{aligned}L(\xi_{q+1})\eta_q &= [\xi_q,\eta_q] + i(\eta)D\xi_{q+1}\\ &= \{\xi_{q+1},\eta_{q+1}\} + i(\xi)D\eta_{q+1}\,,\end{aligned} \tag{A.1}$$

the expression at hand is rewritten as

$$\begin{aligned}&i(\zeta)L(j_1([\eta_{q+1},\xi_{q+1}]))\chi_q + \{\eta_{q+1}, i(\zeta)D\xi_{q+2}\}\\ &\quad + i(\eta)D(i(\zeta)D\xi_{q+2}) - i([\eta,\zeta])D\xi_{q+1} - \{\xi_{q+1}, i(\zeta)D\eta_{q+2}\}\\ &\quad - i(\xi)D(i(\zeta)D\eta_{q+2}) + i([\xi,\zeta])D\eta_{q+1}\,.\end{aligned}$$

By a direct computation [20, p. 383] giving

$$i(\eta)D(i(\zeta)D\xi_{q+2}) - i([\eta,\zeta])D\xi_{q+1} = i(\zeta)D(i(\eta)D\xi_{q+2})\,,$$

combined with the following technical formula [22, p. 209]:

$$i(\zeta)D\{\eta_{q+1},\xi_{q+1}\} = \{i(\zeta)D\eta_{q+1},\xi_q\} + \{\eta_q, i(\zeta)D\xi_{q+1}\},$$

we are left with

$$\begin{aligned} &i(\zeta)L(j_1([\eta_{q+1},\xi_{q+1}]))\chi_q \\ &\quad + i(\zeta)D\big(\{\eta_{q+2},\xi_{q+2}\} + i(\eta)D\xi_{q+2} - i(\xi)D\eta_{q+2}\big) \\ &\quad = i(\zeta)\big(L(j_1([\eta_{q+1},\xi_{q+1}]))\chi_q + D([\eta_{q+1},\xi_{q+1}])\big). \end{aligned}$$

This completes the proof.

Appendix B

In this appendix we put together the main ingredients and equations that we will need in order to make the comparison with the Spencer sequence. For the sake of convenience the notation introduced in ref. [11] is kept.

(i) The BRS transformation of the Beltrami differential μ is defined through a nilpotent s-operation by

$$s\mu = \bar{\partial}C + C\partial\mu - \mu\partial C, \quad sC = C\partial C, \quad s^2 = 0, \tag{B.1}$$

where C is the ghost field as introduced in ref. [30], and the pair (μ, C) defines the chiral splitting. Of course there are the complex conjugate expressions, and we restrict ourselves to the sector generated by (μ, C).

(ii) As for a Yang–Mills type theory the construction of the bigraded algebra is achieved by defining a nilpotent operation $\tilde{d} = d + s$ with the algebraic connection $\tilde{\mu} = \mathrm{d}z + \mu\,\mathrm{d}\bar{z} + C$. The "russian formula", which is given by the vanishing of the two-form

$$\tilde{d}\tilde{\mu} - \tilde{\mu}\partial\tilde{\mu} = 0, \tag{B.2}$$

is a compact way for writing equivalently the BRS transformations (B.1).

(iii) The authors of ref. [11] consider the maximal proper Lie subalgebra w_2 of the Virasoro algebra

$$w_2: \ [L_m, L_n] = (m-n)L_{m+n}, \quad m, n \geq -1. \tag{B.3}$$

We associate a "gauge field", $\widetilde{M}^n$, a one-form, with each generator L_n, $n \geq -1$. Then the curvature [#9] $\widetilde{F}^n$ for $n \geq -1$ is written as

$$\widetilde{F}^n = \tilde{d}\widetilde{M}^n + \sum_{k,l\geq -1} \tfrac{1}{2}(k-l)\widetilde{M}^k\widetilde{M}^l\delta^n_{k+l}, \tag{B.4}$$

[#9] This formula, kindly communicated by R. Grimm, is rewritten in a slighty different way from that worked out in ref. [11] and takes into account the misprint in formula (6) of ref. [11].

where the structure constants $(k-l)$ of the commutation relations (B.3) are exhibited. According to the bigrading each one-form $\widetilde{M}^n$ will be parametrized as follows:

$$\widetilde{M}^n \equiv \vartheta^n + \Upsilon^n = \vartheta^n_z \, dz + \vartheta^n_{\bar{z}} \, d\bar{z} + \Upsilon^n , \tag{B.5}$$

where the ordinary one-form ϑ^n carries a null ghost number and Υ^n is a zero-form with ghost number one.

Imposing the vanishing curvature condition $\widetilde{F}^n = 0$ at each level, the resolution step by step of this hierarchy of equations exhibits a remarkable structure property.

Let us perform the first two steps for solving these curvature equations. The main point is to fix the initial data, that is, $\widetilde{M}^{-1} \equiv \widetilde{\mu}$. This is the *choice* made in ref. [11]. Then let us start by taking $n = -1$ and look for $\widetilde{M}^0$. By a straitforward computation the decomposition with respect to the ghost grading yields

$$\text{ghost}\sharp = 0 \qquad \partial\mu + \vartheta^0_z\mu - \vartheta^0_{\bar{z}} = 0 , \tag{B.6a}$$

$$\text{ghost}\sharp = 1 \qquad \begin{cases} s\mu - \bar{\partial}C + \mu\Upsilon^0 - \vartheta^0_{\bar{z}}C = 0 , \\ \Upsilon^0 - \partial C - \vartheta^0_z C = 0 , \end{cases} \tag{B.6b}$$

$$\text{ghost}\sharp = 2 \qquad sC = C\Upsilon^0 . \tag{B.6c}$$

Let us consider the last equation (B.6c). Thanks to the fact that $C^2 = 0$ we have in complete generality $\Upsilon^0 = \partial C + \gamma^0 C$, where γ^0 is a (commuting) complex number of null ghost number. Then going up trough the equations we obtain the general solution

$$\widetilde{M}^0 = \gamma^0 \widetilde{M}^{-1} + \partial \widetilde{M}^{-1} . \tag{B.7}$$

Similarly for $n = 0$ the part of the ghost number is written as

$$s\Upsilon^0 = 2C\Upsilon^1 \Rightarrow \Upsilon^1 = \tfrac{1}{2}\partial\Upsilon^0 + \gamma^1 C , \tag{B.8}$$

and the ambiguity propagates through the equations of lower ghost number.

Appendix C

The step to be discussed is formula (4.11) in the text. In this appendix, it is shown how the formal properties of the prolongations R_{q+1} will actually determine the algebraic commutation relations of the w_2 (infinite) Lie algebra (B.3).

Let us first recall the general formula of the algebraic bracket (2.35) on $J_{q+1}(T)$ for $0 \le |\mu| \le q$,

$$\{\xi_{q+1}, \eta_{q+1}\}^k_\mu = \sum_{|\lambda|=0}^{|\mu|} \sum_{|\nu|=0}^{|\mu|} \frac{(\lambda+\nu)!}{\lambda!\,\nu!} (\xi^r_\lambda \eta^k_{\nu+1_r} - \eta^r_\lambda \xi^k_{\nu+1_r}) \, \delta^\mu_{\lambda+\nu} , \tag{C.1}$$

where the combinatorial factor in the multi-index notation stands for

$$\prod_{i=1}^{n} \frac{(\lambda_i + \nu_i)!}{\lambda_i!\,\nu_i!} \delta^{\mu_i}_{\lambda_i+\nu_i} .$$

These components (C.1) can be thought of as the successive derivatives up to order q in the Taylor expansion of the bracket $[\xi, \eta]$.

According to the complex decomposition $R_{q+1} = R^{1,0}_{q+1} \oplus R^{0,1}_{q+1}$ (which implies $\xi_{q+1} = \xi^z_{q+1} \oplus \xi^{\bar z}_{q+1}$) the algebraic bracket (C.1) on $J_{q+1}(T)$ will be restricted to

$$R^{1,0}_{q+1} \subset J_{q+1}(T^{1,0}) : \ \xi^z_{\alpha+1_{\bar z}} = 0 , \quad 0 \le |\alpha| \le q , \tag{C.2}$$

and for $\xi^z_{q+1}, \eta^z_{q+1} \in R^{1,0}_{q+1}$, with $0 \le |\alpha| \le q$, it takes the form [#10]

$$\begin{aligned} \{\xi^z_{q+1}, \eta^z_{q+1}\}_\alpha &= \{\xi_{q+1}, \eta_{q+1}\}^z_\alpha \\ &= \sum_{|\lambda|=0}^{|\alpha|} \sum_{|\nu|=0}^{|\alpha|} \frac{(|\lambda| + |\nu|)!}{|\lambda|!\,|\nu|!} (\xi^z_\lambda \eta^z_{\nu+1_z} - \eta^z_\lambda \xi^z_{\nu+1_z}) \delta^\alpha_{\lambda+\nu} . \end{aligned} \tag{C.3}$$

Let us perform the changes in $R^{1,0}_{q+1}$,

$$\begin{aligned} &\xi^z_\lambda \equiv |\lambda|!\,\hat\xi^z_\lambda , && \text{for } 0 \le |\lambda| \le q + 1 , \\ &\{\xi_{q+1}, \eta_{q+1}\}^z_\alpha \equiv |\alpha|!\,\widehat{\{\xi_{q+1}, \eta_{q+1}\}}^z_\alpha , && \text{for } |\alpha| \ge 1 . \end{aligned} \tag{C.4}$$

Thus (C.3) takes the form

$$\widehat{\{\xi_{q+1}, \eta_{q+1}\}}^z_\alpha = \sum_{|\lambda|=0}^{|\alpha|} \sum_{|\nu|=0}^{|\alpha|} (|\nu| + 1)(\hat\xi^z_\lambda \hat\eta^z_{\nu+1_z} - \hat\eta^z_\lambda \hat\xi^z_{\nu+1_z}) \delta^\alpha_{\lambda+\nu} . \tag{C.5}$$

Inverting the role of $\hat\xi$ and $\hat\eta$ in (C.5) and substracting we are left with

$$\begin{aligned} \widehat{\{\xi_{q+1}, \eta_{q+1}\}}^z_\alpha &= (|\alpha| + 1)(\hat\xi^z \hat\eta^z_{\alpha+1_z} - \hat\eta^z \hat\xi^z_{\alpha+1_z}) \\ &+ \sum_{|\lambda|=1}^{|\mu|} \sum_{|\nu|=0}^{|\mu|-1} \tfrac{1}{2}(|\nu| + 1 - |\lambda|)(\hat\xi^z_\lambda \hat\eta^z_{\nu+1_z} - \hat\eta^z_\lambda \hat\xi^z_{\nu+1_z}) \delta^\alpha_{\lambda+\nu} . \end{aligned} \tag{C.6}$$

If we set, for any pair of positive integers (m, n)

$$\begin{aligned} L_m &: (z, \bar z) \mapsto (z, \bar z, 0, ..., , 0, -\hat\xi^z_{m+1}(z, \bar z), 0, ..., 0) , \\ L_n &: (z, \bar z) \mapsto (z, \bar z, 0, ..., , 0, -\hat\eta^z_{n+1}(z, \bar z), 0, ..., 0) , \end{aligned} \tag{C.7}$$

[#10] Thanks to formula (C.2) the multi-indices will only be considered with respect to the z-component. This remark explains the occurrence of the multi-index lenghts in the combinatorial factor of formula (C.3).

as particular sections of $\mathrm{R}_q^{1,0}$ for $q \geq m + n + 1$, where by virtue of (C.2) the integers m, n only count the z-indices, it turns out that the bracket of sections (C.6) reads

$$\{L_m, L_n\}_\alpha = -(m-n)\hat{\xi}^z_{m+1}\hat{\eta}^z_{n+1}\delta^\alpha_{m+n+1} = (m-n)L_{m+n} \,. \tag{C.8}$$

So let us denote the projective limit over the integers $n \geq -1$ by $\mathrm{R}_\infty^{1,0} = \mathrm{pr}\lim \mathrm{R}_{n+1}^{1,0}$ and define the bracket on $\mathrm{R}_\infty^{1,0}$ from (C.8) by

$$[L_m, L_n] \equiv \{L_m, L_n\}_{m+n+1} = (m-n)L_{m+n} \,, \quad m, n \geq -1 \,, \tag{C.9}$$

where the algebraic bracket is computed on $\mathrm{R}_q^{1,0}$, for $q \geq m + n + 1$. We thus recognize at once the commutation relations of the maximal Lie subalgebra w_2 of the Virasoro algebra, formula (B.3).

Let us add some more words about the way to recover the notion of a Lie algebra. First let us recall the notation $\mathcal{R}_q = \mathcal{R}_q(X, X)$ and fix a point $(z_0, \bar{z}_0) \in X$. The isotropy Lie group $G_q = \mathcal{R}_q((z_0, \bar{z}_0), (z_0, \bar{z}_0))$ of $\mathcal{R}_q$, see section 2, made by jets with the same source and target has as projective limit an infinite dimensional Lie group $G_\infty = \mathrm{pr}\lim G_q$ with an infinite Lie algebra $\mathcal{G} = \mathrm{pr}\lim \mathcal{G}_q$, where $\mathcal{G}_q$ is the Lie algebra of $G_q^{1,0}$. Since the Cauchy–Riemann system is transitive, the construction is independent of the choice of the point $(z_0, \bar{z}_0)$. So the bundle $\mathrm{R}^0_\infty(X, (z_0, \bar{z}_0)) = \mathrm{pr}\lim \mathrm{R}^0_q(X, (z_0, \bar{z}_0))$ is a bundle of Lie algebras, that is, over each point of X the fiber is the isotropy part of the infinite Lie algebra w_2.

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